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Anthony Emeka Ohazulike

Road Pricing Mechanisms

A Game Theoretic and Multi-level Approach

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ROAD PRICING MECHANISMS

A GAME THEORETIC AND MULTI-LEVEL APPROACH

PROEFSCHRIFT

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ANTHONY EMEKA OHAZULIKE

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Dedication

I dedicate this work to my entire family. To my beloved wife, a gift from God, thank you for being with me and for your love. To my paternal grandmother of blessed memory, a woman filled with love and kindness, thank you for loving me. To my maternal grandmother an achiever, a woman of an amazing strength who raised my mother, thank you for always being there for us. To my father who taught me the value of hard work, that I can achieve anything in life through hard work, thank you for your inspirational words. To my amazing mother who taught me the value of prayer that shaped my life, thank you so much for your love. To my sisters Ozzi, Koko, Oluu and Nneo, for your warmth love, I could not have asked God for a better family. To Daddy Jerome, a man with a heart full of love and kindness, thank you for your unconditional love and for believing in me.

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I know I have forgotten a lot of names, my head is so full now, please know that you are all in my heart, and thank you a lot for being part of my story.

- Anthony E. Ohazulike
Enschede, November 2013

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Chapter 1

Introduction

1.1 Background

Traffic externalities such as congestion, air pollution, unacceptably high noise levels, accident rates and road maintenance costs are increasingly becoming problematic in most countries. Over the past years, vehicle ownership has increased tremendously. It has been realized that the social cost of owning and driving a vehicle does not only include the purchase, fuel, and maintenance fees, but also the cost of man hour loss due to congestion and road maintenance. Costs of health issues resulting from accidents, inhalation of poisonous compounds emitted from vehicle exhaust pipes, and exposure to high noise level from vehicles add to the welfare loss. Due to financial, geographical, and political limitations, and the fact that even the expansion of the existing infrastructure may not lead to efficient use of transportation networks [12], it is envisaged that road pricing can be used as a tool to achieve a more efficient use of the existing infrastructure. However, until now, researchers have mostly focused on congestion pricing neglecting the overall effect of such practice on the entire network system as well as on the other traffic externalities. Most real life optimization problems require the "simultaneous optimization" of more than one objective. This is because many real life problems concern many different objectives. In most cases, these objectives are in conflict with each other and may or may not be equally important. Tolls will be employed to influence the behaviour of network users so as to achieve a desired network routing. Our motivation for this research further stems from the 'unrealistic but easy' assumption that toll setting is the role of just one actor, for example in Joksimovic [27]. However, this is not the case in general. Specifically, if different stakeholders are allowed to place (or at least influence) tolls in the network, it becomes clear that the stakeholders may have conflicting objectives which will lead to conflicting toll proposals. For example, the insurance companies may like to set tolls to minimize road accidents, whereas the ministry of economics may be interested in minimizing man-hour loss due to traffic congestion so as to boost productivity. On the other hand, a toll proposal that increases speed, thus reducing congestion, may lead to an outright increase in traffic emission, noise and safety related issues. Our main interest is to analyse from a game theoretical perspective the situation where these actors seek various (usually conflicting) road pricing schemes in order to support their different goals. Traditionally, the computation of tolls that minimizes a single objective is formulated as a bi-level optimization problem. *Since game theory can beautifully describe human behaviour, it is well suited to investigate the behaviour of all the players, and particularly, when and how a "cooperative" solution concept in the form of a common road pricing scheme can be found for the actors.* We also have to bear in mind that cooperation among actors may be to the detriment of the "poor" road users. Thus, our task includes developing a pricing scheme that will leave everyone (both actors and road users and even the central government) simultaneously contented. No doubt, this may involve more than a two-level optimization approach, so the use of rigorous and yet simple mathematical tools will be inevitable. Finally, we expect that this study leads to more realistic and convincing "fair tolling schemes" which can also be used in the future Dutch tolling system (Anders Betalen voor Mobiliteit or ABvM). These models are analysed for a static traffic assignment, and be extended to time dependent traffic assignment.

1.2 Problem statement

Though the need for road pricing is well understood, its implementation has suffered setbacks for political reasons and poor levels of acceptance. The reasons for the setbacks are mostly because the proposed schemes are not considered 'fair' enough from the users' and stakeholders' perspectives Schaller [54]. Users and stakeholders feel that their interests are either not represented in the objectives or do not have equal weight as the interests of other stakeholders involved in the decision making process. Moreover, if the factors considered in a pricing scheme are ill-defined, it may not survive intelligent arguments both from political and academic arenas. It is in view of the aforementioned setbacks that this research intends to study the road pricing problem from a game theoretical perspective. Since road pricing has a lot of potentials, especially when it comes to alleviating and/or reducing traffic externalities, it is obvious that there is a need for a road pricing scheme that incorporates fairness and equity issues leaving all participating actors contented, and thus improving the acceptance level of the scheme.

1.3 Research objectives, scope and questions

1.3.1 Research objectives

Our overall objective of this thesis is to develop a road pricing scheme that is fair or at least perceived to be fair by all stakeholders involved. With the scheme developed, we hope that road pricing potentials will be fully harnessed with little political and acceptance issues. We intend to analyse mathematically how and when optimal fair tolls can be achieved.

1.3.2 Research scope

The thesis focuses on deriving a general fair tolling scheme that is optimal for the transportation system and the society. To achieve this goal, our research calls for the interplay of three different fields in applied mathematics and traffic engineering, namely: (1) **Game theory**, in which mathematical attempts are made to capture behaviour in strategic situations, where a player's success in making choices depends on the choices of other players; (2) **Multi-level/ multi-objective optimization** which involves solving more than one-level optimization problem and where each level may comprise more than one objective; and finally

(3) **Traffic engineering** which employs engineering techniques (mainly on research and construction of the immobile infrastructure) to achieve the safe and efficient movement of people and goods. These infrastructures not only include roads, railway tracks and bridges, but also the use of dynamic elements such as traffic signals and lights, detectors, sensors and tolls. The theory of games mentioned earlier will be used to extensively analyse behaviour of the rational road users and the stakeholders.

1.3.3 Research questions

Owing to the problem statement in subsection 1.2, the research questions as discussed in this thesis fall into five major categories which do not have clear demarcations. These are set out below.

1. Under static traffic assignment (STA), the thesis addresses the following questions:
   - What happens when stakeholders do not cooperate in toll setting?
   - Under which conditions can the existence of a Nash equilibrium (NE) be guaranteed?
   - Can we design a mechanism that induces a Nash equilibrium between the actors?
   - When and how can a cooperative solution concept in the form of a common road pricing scheme be found?
   - If the stakeholders agree to cooperate, how would they share the benefits?
   - Can we design a mechanism that induces a cooperative outcome on otherwise non-cooperative actors, and thus achieve the system optimum or any other prescribed state within the system?
   - Which coalitions among the stakeholders are likely to be formed in a cooperative concept?
   - What can we say about the various classical solution concepts from cooperative game theory, such as core, nucleolus and bargaining sets?
2. Equity issues:

   Can we design a tolling scheme such that:
   - People do not pay “unnecessarily high” tolls because of where they live or work.
   - Flat tolls or user-specific tolls: which is more acceptable and to whom?
   - OD-based tolls or link-based tolls: which is better from both the system’s and users’ perspectives?
   - Finally, can we find a tolling scheme that leaves every player (including the road users) contented?
3. Implementation and practical application of our model
   - How does the model apply to a realistic network.
4. Model extension
   - In which other domains might our models be applicable?

5. What are the policy implications of the study?
    - What can the government, stakeholders, and road users learn from it?
    - How feasible are the models?

1.4 Research approach

Owing to the nature of the task at hand, the problem will be formulated as a multi-level multi-objective optimization problem with various players handling different objectives at various levels. Firstly, we will model the stakeholders at one level and the road users at a lower level. The stakeholders will influence the behaviour of the road users with the aid of road tolls. In contrast to the traditional way of modelling road pricing where a leader controls the road users using tolls in order to achieve a cumulative benefit for the stakeholders or the society, the proposed model allows the stakeholders to compete for optimal link flows and tolls due to their conflicting objectives. Each actor will always try to set a toll such that the cumulative link toll vector will profit him and him alone. We will study the effect of the actors' selfish toll setting on themselves as well as the effects on the road users. This will be modelled as an equilibrium problem with equilibrium constraints (EPEC). The stability of the toll setting game between the actors will be investigated under the Nash equilibrium (NE) concept from game theory. With the results we have, we then go on to develop a road pricing model for non-cooperative actors. If NE does not exist, then we will be challenged to design a mechanism that will enforce NE among actors since only then will the game terminate. Cooperation among some stakeholders over the level of toll to be set on a given road segment will also be included in the stakeholders' problem.

Since a grand coalition that will likely increase the system efficiency may not naturally be formed between the actors, we will design a mechanism and/or profit sharing scheme that can induce a cooperative outcome on the actors. With this in mind, we intend to introduce a higher authority (say the government/grand leader) who designs these mechanisms and 'imposes' them indirectly on the actors taking part in the road pricing game. This will again lead to a road pricing model for cooperative actors. Answers to research questions such as which mechanism design can induce Nash equilibrium and/or cooperation among the actors, possible coalitions and the study of incentives for cooperation will be clearly demonstrated in the model. Using the developed models, we will make detailed and striking comparisons between the non-cooperative and cooperative pricing approaches. We expect this comparison to lead to a suggestion for the most effective road pricing scheme(s). This suggestion will be confirmed by testing the models using numerical examples. Our study will simultaneously deal with fixed and elastic demands. The models developed under static traffic assignment will then be extended to time dependent traffic assignment owing to its practical flavour. We will carry out sensitivity tests to determine how link tolls change with network structure, demand and social welfare. We will then investigate various classical solution concepts of cooperative and non-cooperative game theory. These concepts include: Nash equilibrium, core, bargaining sets, cost sharing formulae and stability of solutions among others. Other tolling schemes such as OD-based scheme will be studied and compared with the link-based scheme. In the end, we will show that our models are not, in fact, limited to a transportation field, but can be applied in other interesting fields to solve equity, fairness and optimization

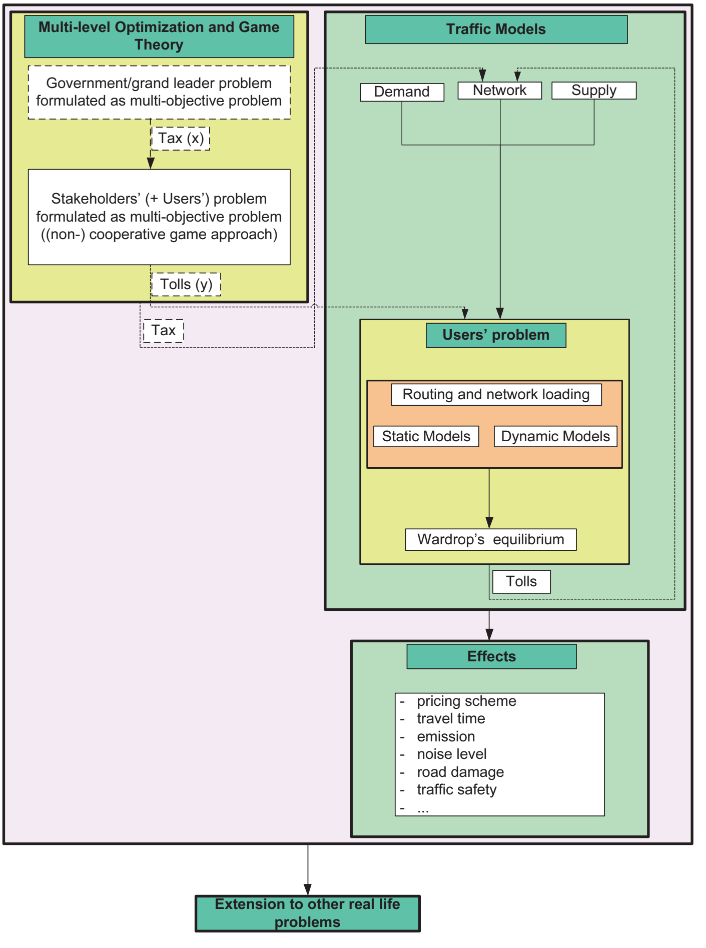


Figure 1.1: Theoretical framework

issues. The Figure above describes the modelling idea of the multi-level solution approach to the problem.

Figure 1.1 above describes our game theoretical approach to solving the multi-objective multi-stakeholder road pricing problem where the stakeholders compete for a desired toll pattern to optimize their individual objectives. The stakeholders' tolls are used to steer road users to a desired traffic flow pattern. As a mechanism to control the behaviour of stakeholders, the Grand leader (at the highest level of the multi-level system) uses tax mechanism to steer the stakeholders to choose a set of toll patterns desired by the system or by the Grand leader. As shown in the flow diagram above, the tolls as well as the taxes are returned back into the transportation system so as not increase societal costs.

The problem setting is as follows: we take that the Grand leader (GL) knows that the users will react according to Wardrop's equilibrium on perceiving the tolls from the stakeholders, the GL also knows that the stakeholders will toll the network in order to optimize their individual objectives F_i, so knowing these facts, he solves the system problem, levying taxes (x) on the stakeholders to steer them towards a toll pattern that will in turn steer the users to a system desired flow pattern z. At the second level, the stakeholders also know that the users behave according to Wardrop's equilibrium, so they choose their tolls (y) in a fashion that steer the road users to an equilibrium flow pattern that satisfies their individual interests. The lowest level represents the road users who travel according to Wardrop's equilibrium with respect to total travel costs (including the travel time costs, tolls, and so on).

Figure 1.1 gives a schematic overview of the tolling model and can mathematically be described as follows:

$$
\begin{array}{ll}
\min\limits_{x,y,z} \left(F_1\left(x,y,z\right),\, F_2\left(x,y,z\right), \cdots\, F_k\left(x,y,z\right),\, f(y,z)\right) & [Grand\,Leader] \\
\quad s.t & \\
\quad \tilde{G}\left(x,y,z\right) \leq 0 & \\
\quad \min\limits_{y,z} \left(F_1\left(x,y,z\right)\right), \min\limits_{y,z} \left(F_2\left(x,y,z\right)\right), \cdots, \min\limits_{y,z} \left(F_k\left(x,y,z\right)\right) & [Stakeholders] \\
\qquad s.t & \\
\qquad G\left(x,y,z\right) \leq 0 & \\
\qquad\qquad \min\limits_{z} f\left(y,z\right)) & [Users] \\
\qquad\qquad s.t & \\
\qquad\qquad g\left(\, y,z\right) \leq 0 &
\end{array}
$$

Where $x \in \mathbb{R}^{|\mathbb{K}\times\mathbb{A}|}$ are the Grand leader variables (the taxes), $y \in \mathbb{R}^{|\mathbb{A}|}$ are the stakeholders variables (the tolls), and $z \in \mathbb{R}^{|\mathbb{A}|}$ are the users' variables (the flows). F_i is actor *k's* objective function, and f is the users' objective function. $\tilde{G}\left(x,y,z\right)$, $G\left(x,y,z\right)$ and $g\left(y,z\right)$ are constraints associated with the GL, the stakeholders and the users respectively. Observe that each level is a full optimization program. Notice also that both the stakeholders' and the users' problems are fully represented in the *GL's* problem. Figure 1.1

1.5 Relevance

1.5.1 Introduction

The research in this thesis falls within the road pricing subject of transportation economics, concatenating three different fields in applied mathematics and traffic engineering, namely: (1) **Game theory**, (2) **Multilevel/ multi-objective optimization** and (3) **Traffic engineering**. Furthermore, road pricing falls within the scope of 'strategic modelling for sustainable development'. Traffic engineering involves the study of traffic flows, measures, indicators, infrastructures and traffic externalities. The objective of these studies is to understand and improve the overall network efficiency. Road pricing has proven itself over time as one of the tools to optimize the use of transportation networks. We have also mentioned that transportation policies are often determined by multiple stakeholders.

Sometimes, these stakeholders have different interests and objectives, and selfish actions may lead to a bizarre network situation. Similarly, an optimal traffic setting (e.g. traffic light setting) at one road junction may lead to a poor traffic flow at several other junctions, which may result in a bizarre network effect. Our problem, therefore, is to design the traffic light in one junction such that every other junction as well as the entire network operates efficiently. However, we are not going to design a model for traffic signals, but a road pricing scheme that leaves every player contented.

1.5.2 Scientific relevance

As for scientific relevancy, the research will hopefully shed more light on the multi-level optimization approach to many multi-objective problems. It will explain from a game theoretical perspective how different entities with different (conflicting) objectives can be modelled such that every entity is contented. Further, it will demonstrate a mechanism that can be used to lure actors of conflicting interest to a common (system optimum) interest. This can be extended to a wide range of scientific fields as we demonstrate in this thesis. The study will also shed light on analytical models for both static and dynamic road pricing schemes. The study describes various road pricing schemes ranging from existing schemes to brand new schemes.

1.5.3 Societal relevance

For the society, the research represents good news since it considers the effect of traffic on, for example, common road users, equity issues and societal welfare. Implementation of the models developed will also ensure that the stakeholders do not 'enrich' themselves to the detriment of road users.

For the central government, we will provide a mechanism that leads to a cooperative outcome among generally non-cooperative stakeholders and/or regional governments.

For the regional governments or the stakeholders, the outcome of the research will aim to develop tolling schemes that would make road users and stakeholders contented.

1.6 Thesis outline

We present the summary of this thesis in the form of a flowchart enabling the reader to navigate to chapters of interests.

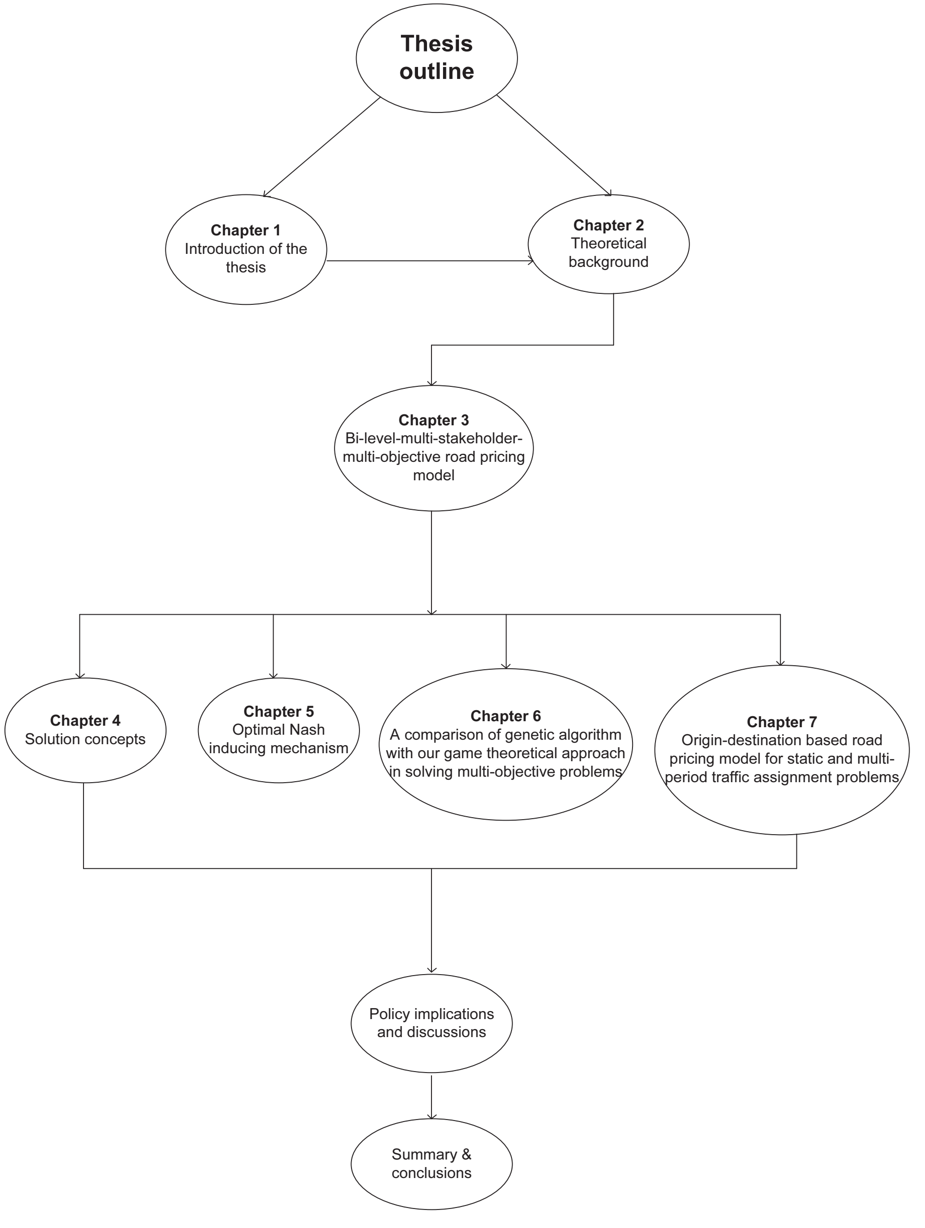


Figure 1.2: The outline of the thesis

Chapter 2

Theoretical background

2.1 Developments in road pricing

2.1.1 Introduction

Economists found that when a resource that is vital and scarce is free or underpriced, then demand for such a resource will outstrip the supply, resulting in shortages. This phenomenon is readily seen in the transportation sector. When the demand or number of vehicles using a certain road exceeds the road's capacity, then congestion begins to build up. This using up of road capacity mostly occurs during the so called peak hours. In 2006, it was estimated that there were 41,118 traffic jams, and approximately 60 million vehicles lost hours on Dutch highways [10, 6, 5]. It is estimated that the total amount of vehicles that lost hours on the secondary road network is even higher [20]. In the most populous and industrialized African city, Lagos, Nigeria, with a population of over 14 million people, it is estimated that each resident on average loses approximately three hours every day to road congestion (personal experience). The negative effects of congestion range from time and man-hour losses to damages to pavements, environment and residents in the urban areas. Owing to these undesirable effects and for the fact that road expansions may be infeasible, and even more be counter productive (due to Braess paradox) [12], experts propose the use of road pricing to tackle these problems. As road is a valuable and scarce resource, they suggest that it ought to be rationed by a pricing mechanism. Road users should pay for using the road network to make correct allocation decisions between transport and other activities. On the other hand, imposing tolls on (certain) roads will make users change their planned routes, and by so doing, traffic is more 'evenly' distributed throughout the network in and over time. They argued that this reduces the entire network travel time. With the advent of electronic road pricing techniques, it is now easier than ever to implement road pricing since cars no longer have to stop before being charged. This will ensure that road pricing does not create unnecessary congestion. Others [1, 22] propose the use of vehicle tax and differential parking charges to combat congestion. They argued that imposing taxes on vehicles will discourage people from buying cars. By charging users of certain roads to specific parking areas, which they argue, will reduce the number of cars entering a congested area while not interfering with business activities and shopping. For an initial short amount of time, users are charged relatively small amounts, and then for longer periods, they are charged more. Thus, users who need short term parking benefit from such a parking scheme while others are encouraged to use public transportation and commuting. In fact, the truth is that the benefits of owning a car outweigh the taxes, and the subsidies received by employees from their employers for transportation fares cushion the effect of parking charges. These reasons make it impossible to tackle congestion problems with such taxing schemes proposed in [1, 22]. Furthermore, parking fees do not depend on the traffic volume or distance travelled, neither do they depend on the environmental characteristics of the vehicle. Traffic in transit through a congested area is not affected by parking fees at all. These show that parking fees are not efficient ways to battle traffic externalities. The interested reader is referred to [39] for an early survey of such schemes. Before we continue with different road pricing schemes, let us first see a few selected implementations of road pricing in the world today.

2.1.2 Applications of Road Pricing

In this section, we review some places where road pricing schemes have been implemented successfully, the advantages and disadvantages and where they have failed.

Singapore in 1975 introduced the Area Licensing scheme (ALS) making it the first country to design and implement a practical (low-tech) congestion pricing. This was later replaced by Electronic Road Pricing (ERP) in 1998. The aim was to check traffic (at peak periods) into the Central Business District, so that congestion is minimised. The tolls would vary according to average speed on the network.

Norwegian cities of Bergen, Oslo and Trondheim, in 1986, 1990 and 1991 respectively, introduced cordon pricing scheme to raise revenue for financing road projects and to small extent, public transport. Though the scheme was not originally designed to reduce traffic, some impact on travel behaviour and traffic volumes were noticed. One drawback of the Trondheim toll ring as a financing mechanism is that about one-third of the region's drivers live inside the ring and therefore, rarely pay charges, yet they benefit from some of the road improvements.

Autoroute A1 is an express way connecting Paris to Lille, about 200 km to the north. Vehicles receive a ticket upon entering the express way and pay at a toll booth upon exiting, an amount depending on the length of the trip. The A1 is subject to heavy traffic near Paris on Sunday afternoons and evenings. In April 1992, after a period of extensive public consultation and publicity, this congestion problem was confronted by implementing a time varying toll scheme for Sundays only. A special 'red tariff' is charged during the Sunday peak period (16:30–20:30), with toll rates 25 to 56 percent higher than the normal toll. Before and after the peak, there is a 'green tariff' with rates 25 to 56 percent lower than the normal toll. These hours and rates were designed so that total revenues are nearly identical to those collected with the normal tariff. This property was believed essential for public acceptance, which, in fact, has been largely high.

San Diego I-15 Express in 1996 introduced a high occupancy toll (HOT). As a result, the spare capacity of the HOT lanes is now more efficiently used, and moreover, many users are enthusiastic over the scheme and are ready to pay provided they get an enhanced service.

In 1997, 407 Express Toll Road, Toronto started operation. Traffic demand rose from 11,000 to 12,000 cars (for this road) in peak hours, and at the same time, average speed is about double that of the nearby congested public highways.

London, on February 17, 2003, kicked off congestion pricing to reduce traffic congestion, increase journey time reliability and decrease air pollution. The benefits of London's congestion charging include: 15% traffic reduction, 30% congestion reduction, 12% pollution reduction (CO_2, NO_x, PM_{10}), journeys became more reliable, buses significantly gain time reliability, and substantial reduction of road accidents among others.

On January 1, 2005, the German Federal Government introduced distance-related tolling of heavy trucks ($\geq 12\,tons$) using "Autobahns". The technology is based on GPS/GSM in order to have the option to extend tolling to all kinds of roads and vehicles later. The amount of tolls is based on the internal/direct costs caused by heavy trucks - calculated according to a special EU-directive. The net toll-revenue was decided to be used exclusively for the transportation infrastructure - 50 % for the Federal Highways, 50 % for the Federal railways and the inland waterways. Since the inception of the scheme, it has been working without any problems.

Stockholm in Sweden introduced a trial system with 19 toll plazas from 3rd January to 31st July 2006. After successful trials, the system was continued from fall 2007.

On 2nd January 2008, Milan became the first metropolitan area in Italy to introduce a congestion charge for the city centre. The scheme aims at reducing congestion and air pollution.

On 1st October 2008, an extension of the current charging zone around Oslo was established to cover the suburbia Baerum west of Oslo. The new zone is operated separately, which means that vehicles going to Oslo from the west will be charged twice. The system is based on the existing system in Oslo, which is fully automated. The motivation for the new charging zone is to provide capital for the operation of the public transport in Oslo and Akershus the next twenty years. Moreover, it will provide capital for expansion of the west corridor out of the Norwegian capital.

Though road pricing has successfully been implemented in some countries, it has failed in some countries due to poor public acceptance among other factors.

Hong Kong, in 1986 proposed a congestion pricing scheme which was never implemented due to some factors, which include improved traffic flows and the mid 1980's recession in Hong Kong.

Edinburgh city in 2002 decided to carry out extensive public hearings during 2003, followed by a referendum. The referendum was only meant to be guiding. The public hearings lead to some adjustments of the design of the road pricing system. The referendum took place in February 2005, and the result was that 74% voted "no" to introducing road pricing while only 26% voted "yes". This resulted in the City Council's decision to immediately drop all plans to introduce road pricing. Most of the planned investments will still be carried out but will now be financed by ordinary taxes.

Though Trondheim road pricing reduced the inbound traffic by up to 10%, the charging scheme as mentioned earlier was not operated with the intention of reducing congestion but was implemented over a 15 year period in order to gain funds largely for road investments. This period ended in 2005 and as such, Trondheim is the first city ever to stop collecting tolls. Proposals are currently under debate however, even though traffic conditions have not been viewed as problematic.

New York congestion pricing was a proposed traffic congestion fee for vehicles travelling into or within the Manhattan central business district of New York City. The congestion pricing charge was one component of New York City Mayor Michael Bloomberg's plan to improve the city's future environmental sustainability while planning for population growth, entitled PlaNYC 2030: A Greener, Greater New York. If approved and implemented, it would have been the first such a fee scheme enacted in the United States. The deadline to approve the plan by the State Assembly was April 7, 2008, for the city to be eligible to receive US$ 354 million in federal assistance for traffic congestion relief and mass transit improvements. On April 7, 2008, after a closed-door meeting, the Democratic Conference of the State Assembly decided not to vote on the proposal, "...the opposition was so overwhelming, ...that he would not hold an open vote of the full Assembly," Sheldon Silver, the Assembly Speaker said. Afterwards, the US Department of Transport (UDOT) announced that they would seek to allocate those funds to relieve traffic congestion in other cities. Chances for the bill to return soon to the State Assembly are considered dim, as long as Sheldon Silver remains the Speaker.

The following countries have plans of implementing road pricing in the near future; The Netherlands, some cities in the USA, Hong Kong, Australia, and other cities in Britain, e.g. Bristol, Leeds, Derby, Edinburgh, Leicester, etc.

Netherlands aimed to start with freight transport in 2011. This would require intensive technical and policy-related cooperation with Belgium, France and Germany. If everything had gone as planned, then passenger cars would have followed a year after the launch of freight transport. The complete system roll-out was scheduled for 2016 and beyond.

Returning to the theoretical background of road pricing models, two types of schemes exist within the field of road pricing, namely, first and second-best pricing schemes.

2.1.3 First-best pricing scheme

First-best pricing is a tolling scheme that leads to the optimal use of the transportation network in terms of system travel costs. It requires that all links be made available for tolling. This scheme is sometimes referred to as marginal cost pricing (MCP) for congestion. In this thesis, we define first-best pricing to include marginal costs for other externalities like air pollution, noise pollution, road damage and accidents. MCP on a congested transportation network dates back to Beckmann et al. [7]. They argue that if there were a way to collect tolls from the users of congested roads at rates that would measure the (delay) cost an average road user inflicts on others, a better use of the highway system would be obtained. They also added that the collected revenue should be invested back into the transportation system or to the society so that the social cost or welfare loss is not increased as a result of such tolls. MCP sets on each link a toll which is equal to the difference between the marginal cost, and the average cost (see subsection 2.1.6 below). Ferrari Paolo [21], and Yang and Huang [77] discuss the first best congestion pricing scheme with capacity and environmental constraints. Bergendorff et al. [8], Hearn and Ramana [23], and Yildirim et al [79] discuss the concept of congestion toll pricing framework using the optimality conditions. Their result shows that, in fact, there exists an infinite number of toll vectors that can achieve the optimal flow pattern in terms of congestion. This leads to a redefinition of first-best congestion pricing to include not only MCP, but also those congestion toll vectors that can be used to achieve the most efficient use of the transportation network with respect to congestion. In this thesis, we derive similar results for multi-objective problems.

2.1.4 Second best pricing scheme

Second-best pricing is a tolling scheme that does not assume all links to be available for tolling and allows additional constraints on the tolls. For reasons ranging from political issues to equity concerns, it may be that some links and some user classes are not to be tolled. Further, for time-dependent tolls (see Chapter 6), it may be that some time intervals (usually the off-peak periods) are toll-free. When this is the case, it may be that the system optimum flow is no longer achievable by tolling. Then the toll vector that achieves the best possible sub-optimal flow is called the *second-best toll* vector. Second-best pricing is gaining more practical grounds for the fact that it may be practically impossible to toll all roads for its poor public acceptance, cost of setting up toll booths, political and technical reasons among others. Key questions to answer in the second-best pricing scheme include: where to levy the toll and how much? Yang and Lam [76] model the second-best pricing with elastic demand as a bi-level programming problem. The upper level represents the system controller (or the decision maker) that determines the tolls that optimize a given system's performance while considering users' route choice behaviour. The lower level represents the users. Since the users will always minimize their perceived cost in their route choice behaviour, it means that the system will eventually 'settle' at user equilibrium. The lower level problem, therefore, translates to finding user equilibrium flows. This bi-level problem setting can be seen as a Stackelberg game where the system controller is the leader, and the network users are the followers.

Verhoef [66, 69] study second best pricing with different model formulations, but with the same bi-level approach as Yang and Lam [76]. Verhoef [66] uses the assumption of existence of set a of relevant paths per OD. Yildirim et al [79] arrive at the same result as Verhoef [Verhoef2000] by assuming the existence of Lagrangian multipliers for the associated *Karush-Kuhn-Tucker (KKT)* conditions. Yang and Lam [76] and Verhoef [66] develop algorithms for solving the second-best pricing problem. May et al. [37] prove the convergence of Verhoef's algorithm when applied to a small network section of the city of Leeds. Lawphongpanich et al. [32] and Yildirim [78] formulate the second-best pricing problem as a mathematical problem with an equilibrium constraint expressed as variational inequality for both fixed and elastic demand. They assume the existence of Lagrangian multipliers. Lawphongpanich et al. [32] further investigate the properties associated with the second-best pricing problem and derived an algorithm for it. Owing to the fact that drivers may differ with respect to the value of time and with respect to marginal impact on others, Verhoef and Small [68] consider second-best congestion pricing with heterogeneous drivers on three links, assuming elastic demand and a continuum values of time.

Verhoef et al [67] look at the road pricing problem from a multidisciplinary perspective. Their work includes the optimal design of road pricing schemes, the behavioural effects that may be induced among individuals and firms, and acceptability of road pricing. Olsder [46, 47] models road pricing as an inverse Stackelberg game. A detailed study of Stackelberg and inverse Stackelberg games with its applications, for example, in the optimal toll design and the energy market liberalization problem can be found in [62]. The Game theoretical analogy of road pricing is also part of their research output. All the models above depend on the Wardropian principles.

Due to the constraint on the tolls, the upper level problem is generally solved simultaneously with the lower level problem. This type of problem falls into a special class of a mathematical problem called the Bi-level Programming Problem (BLPP). The second-best pricing problem is usually formulated as a BLPP known as Mathematical Program with Equilibrium Constraint (MPEC). MPECs are generally hard mathematical problems, as we will see. Most existing models in the literature have common shortcomings; firstly, they fail to explicitly define all the externalities caused by road users, and secondly, they all assume the existence of only one decision maker or system controller, usually the government. Both aspects are dealt with in this thesis.

2.1.5 The Network model

In this section, we present basic notations used in this thesis. We will remind the reader of some of the notations as they are being used, and more notations will be added to this list as we proceed.

Let $G = (N, A)$ be a network, with N the set of all nodes and A the set of (directed) arcs or links in G. A trip starts at an origin node O and ends at a destination node D. We use the following notations:

Table 2.1: Notation table

A	set of all arcs (links) in G
a	index for links
R	set of all paths
r	index for paths (routes)
W	index set of all OD pairs (o_w, d_w), o_w, $d_w \in N$
w	index for OD pairs
f	path flow vector
f_r	flow on path r
v	vector of link flows
v_a	flow on link a
Γ	OD-path incident matrix
Λ	arc-path incident matrix
V	set of feasible link flows
d	travel demand vector
d_w	demand for the w^{th}OD pair
R_w	set of all paths connecting OD pair w

Notation table cont.

$D_w(\lambda_w)$	demand function for the w^{th}OD pair
$B_w(d_w)$	inverse demand function for the w^{th}OD pair
λ_w	least cost to transverse the w^{th}OD pair
K	set of all actors in the road pricing game
$C_a^k(v)$	link cost function for the k^{th}objective
$C^k(v)$	total network cost function for the k^{th}objective with $C^k(v) = \sum_{a \epsilon A} C_a^k(v)$
$C(v)$	vector of network cost functions in G
$Z(v)$	total network cost in $G.\, i.e.$ $Z(v) = \sum_{k \epsilon K} C^k(v)$

2.1.6 Derivation of optimal first-best tolls

In this section, we will derive the optimal road pricing model also known as the first-best tolls. We focus on the elastic demand derivation of the first-best tolls. The derivation will be based on the a single leader or single decision maker. For diversity, we will derive the same optimal tolls in section 3.2.1 using *fixed* demand models.

Decision maker's problem (System Problem - SP)

This is the problem statement that describes the objective of the system controller. The road pricing model with elastic demand involves the simultaneous minimization of controller's objective (for example, travel time) and maximization of user benefit. Therefore, the objective of the system controller to keep the social welfare as high as possible can be stated as follows:

max [Social Welfare (or Economic Benefit)]

s.t

flow and environmental feasibility conditions

The Social Welfare is given by

$$Social\, Welfare = UB - SC$$

where UB is the **User Benefit**, given by

$$UB = \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$B_w(d_w)$ is the inverse demand or benefit function for the OD pair $w \in W$ [77]. Observe that $UB = C$ when the demand is fixed.

The **Social Cost** (SC) is given by

$$SC = C(v)$$

$C(v)$ is the system cost function and can be the system travel time cost, noise cost, emission cost, etcetera.

In this specific derivation, we will take $C^k(v)$ to be the travel time cost $\beta v^T t(v)$, where β is the value of time (VOT), v is the vector of link flows, and $t(v)$ is the vector of link travel time functions.

The system problem **SP** can then be stated mathematically as follows (with the dual variables as indicated at the right of the corresponding constraints):

$$\min_{v,d} Z = \beta v^T t(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma \;\; \text{s.t.} \quad \left.\begin{array}{rcll} v & = & \Lambda f & \psi \\ \Gamma f & = & \bar{d} & \lambda \\ f & \geq & 0 & \rho \\ d & \geq & 0 & \vartheta \end{array}\right\} (FeC\_ED) \quad (2.1)$$
$$\begin{array}{rcll} g(v) & \leq & 0 & \xi \end{array} \Big\} \, Side\, Constraints$$

The first constraint states that the flow v on a link is equal to the sum of all path flows f that pass through this link. The second equation is the flow-OD balance constraint. It states that the sum of flows on all paths originating from origin node p and ending at destination node q for an OD pair pq equals the demand d for the OD pair w. The third and fourth inequalities simply state that the path flows, and OD demands are non-negative. The non-negativity of link flows follows directly from the third constraint. The fifth constraint $g(v) \leq 0$ contains possible side constraints on the link flow vector v. These constraints may be standardization constraints, which may require:

1. That total emission on certain links does not exceed the stipulated emission standard.
2. That total noise level on certain links does not exceed the standard allowed dB(A) level.
3. That total number of cars using certain roads does not exceed a given number or height or weight to preserve the pavement and prevent accidents.

(ψ, λ, ξ, ρ, ϑ) are the *Karush-Kuhn-Tucker (KKT) multipliers* associated with the constraints. For simplicity, we omit the side constraints $g(v)$ in the subsequent formulations. The remaining constraints we call the **feasibility Conditions for Elastic Demand** denoted by *FeC_ED*.

Assumption 1:

- We assume throughout that the link cost (or travel time) function vector $t(v)$ is continuous and satisfies $(t(v) - t(\bar{v}))^T (v - \bar{v}) > 0 \;\; \forall v \neq \bar{v},\, v, \bar{v} \in V$ and all functions $C^k(v)$ are continuous, strictly convex, and strictly monotone (in the sense that $\partial C^k(v)/\partial v_a \geq 0 \; \forall k, a$), and the side constraints $g(v) \leq 0$ (see Eq.(2.1)), if used, are linear.

In the derivation below, we will take that the link travel time functions are separable, i.e. $t_a(v) = t_a(v_a)$ for all links. Later we will generalize to a non-separable link functions which is straightforward.

We now look into the *KKT optimality conditions* of problem (2.1). If we let L be the Lagrangian, and $\bar{v}$, $\bar{d}$ be an optimal solution to program (2.1), then there exist $(\psi, \lambda, \rho, \vartheta)$ such that the following *KKT conditions* hold:

$$
\begin{aligned}
L &= \beta v^T t(v) - \sum_{w\in W}\int_0^{d_w} B_w(\varsigma)d\varsigma + (\Lambda f - v)^T\psi + (d-\Gamma f)^T\lambda - f^T\rho - d^T\vartheta \\
\frac{\partial}{\partial v}L &= \beta\left(t(\bar{v}) + \bar{v}\frac{d}{dv}\left(t(\bar{v})\right)\right) - \psi = 0 \\
&\Rightarrow \beta\left(t_a(\bar{v}_a) + \bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right) - \psi_a = 0 \quad \forall a\in A && (2.2)\\
\frac{\partial}{\partial f}L &= \Lambda^T\psi - \Gamma^T\lambda - \rho = 0 \\
&\Rightarrow \sum_{a\in A}\psi_a\delta_{ar} - \lambda_w - \rho_r = 0 \qquad \forall r\in R_w,\ w\in W && (2.3)\\
\frac{\partial}{\partial\,(d)}L &= \lambda - B(\bar{d}) - \vartheta = 0 \\
&\Rightarrow \lambda_w - B(\bar{d}_w) - \vartheta_w = 0 \qquad \forall w\in W && (2.4)\\
f^T\rho &= 0 \qquad \forall w\in W && (2.5)\\
&\Rightarrow f_r\rho_r = 0 \qquad \forall r\in R \\
d^T\vartheta &= 0 \qquad \forall w\in W && (2.6)\\
&\Rightarrow d_w\vartheta_w = 0 \qquad \forall w\in W \\
\rho,\ \vartheta &\geq 0 \quad i.e. \quad \rho_r, \vartheta_w \geq 0 \qquad \forall r\in R_w,\ w\in W
\end{aligned}
$$

Here $\frac{\partial}{\partial x}f$ is the partial derivative of f with respect to x, and $\Lambda = \delta_{ar}$ is a binary parameter that equals 1 if the link a belongs to the path r and 0 otherwise. Eqs.(2.5) and (2.6) are complementarity constraints.

Note that convexity assumption on the network cost function $\left(C^k(v) = \beta v^T t(v)\right)$ensures that the SP program in Eq.(2.1) has a unique link flow solution $\bar{v}$.

Substituting Eq.(2.2) into Eq.(2.3) yields

$$\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar} = \lambda_w + \rho_r \qquad \forall r\in R_w,\ w\in W \tag{2.7}$$

and Eq.(2.4) into Eq.(2.7) gives

$$\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar} = \left(B(\bar{d}_w) + \vartheta_w\right) + \rho_r \quad \forall r\in R_w,\ w\in W \tag{2.8}$$

since $\vartheta_w,\ \rho_r \geq 0$, $\forall r\in R_w,\ w\in W$, Eq.(2.8) implies

$$\boxed{\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar} \geq B(\bar{d}_w) \quad \forall r\in R_w, \forall w\in W} \tag{2.9}$$

Interpretation: Eq.(2.9) states that the system controller would want the travel cost of every road user to include not only the travel time cost but also the cost of the travel time externality he causes on other users. Furthermore, it states that this total cost should be at least equal to the benefit he (road user) enjoys in making the trip. In fact, the quantity $\beta\bar{v}\nabla\left(t(\bar{v})\right)$ will turn out later to be a feasible first-best link toll vector.

As a result of the flow conservation constraint in Eq.(2.1) and the *KKT conditions*, we derive the following:

$$\left(\text{Recall that} \qquad v = \Lambda f \iff \bar{v}_a = \sum_{w\in W}\sum_{r\in R_w} f_r\delta_{ar} \qquad \forall a \,\epsilon\, A\right)$$

$$\begin{aligned}
\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\bar{v}_a &= \sum_{a\in A}\sum_{w\in W}\sum_{r\in R_w}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right) f_r\delta_{ar} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r \sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r(\lambda_w + \rho_r), \quad \text{using Eqn 2.7} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r\lambda_w + \sum_{w\in W}\sum_{r\in R_w} f_r\rho_r \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r\lambda_w = \sum_{w\in W}\lambda_w\sum_{r\in R_w} f_r, \quad \text{using Eqn 2.5} \\
&= \sum_{w\in W}\lambda_w\bar{d}_w, \quad \text{using Eqn 2.1}
\end{aligned}$$

$$\therefore \sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\bar{v}_a = \sum_{w\in W}\lambda_w\bar{d}_w \tag{2.10}$$

Substituting Eq.(2.4) into Eq.(2.10) yields

$$\begin{aligned}
\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\bar{v}_a &= \sum_{w\in W}\left(B(\bar{d}_w) + \vartheta_w\right)\bar{d}_w \\
&= \sum_{w\in W} B(\bar{d}_w)\bar{d}_w + \sum_{w\in W}\vartheta_w\bar{d}_w \\
&= \sum_{w\in W} B(\bar{d}_w)\bar{d}_w, \quad \text{using Eqn 2.7}
\end{aligned}$$

$$\therefore \sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\bar{v}_a = \sum_{w\in W} B(\bar{d}_w)\bar{d}_w \tag{2.11}$$

Interpretation: Eq.(2.11) states that, for optimal societal benefit, total cost incurred in the system by the travellers should be equal to the total benefit they enjoy.

We thus summarize below the necessary and sufficient condition for the flow pattern$(\bar{v}, \bar{d})$ to be optimal:

$$\boxed{\begin{aligned}
&\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar} \geq B(\bar{d}_w) \qquad \forall r \in R_w,\, w \in W \\
&\sum_{a\in A}\left(\beta t_a(\bar{v}_a) + \beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)\bar{v}_a = \sum_{w\in W} B(\bar{d}_w)\bar{d}_w
\end{aligned}} \tag{2.12}$$

Thus the following are equivalent:

1. *KKT optimality* conditions of problem (2.1)
2. Eq.(2.12) together with the *FeC_ED*

(Road) User Problem - UP

Without loss of generality, we assume that a road user only considers the costs and the benefits he enjoys making a trip. In this way, the only determinant of user's route choice behaviour is the travel costs and benefits of a trip. We use Beckmann's formulation [7] of Wardrop's user equilibrium to describe the users' behaviour mathematically. The formulation shown below is a convex program since the travel time cost functions are assumed to be separable and monotonic, and the benefit function, monotonically increasing:*second-best toll*

$$\begin{aligned} \min_{v,d} \sum_{a\in A} \int_0^{v_a} \beta t_a(u)du - \sum_{w\in W} \int_0^{d_w} B_w(\varsigma)d\varsigma \\ s.t \\ FC\_ED \end{aligned} \tag{2.13}$$

Remark

It is well known that the objective in (2.13) can be mathematically formulated as a variational inequality [77]. It was shown in [77] that a flow pattern (v^*, d^*) is in user equilibrium if and only if it solves the following variational inequality problem:

$$\beta t(v^*)^T (v - v^*) - B\left(d^*\right)^T \left(d - d^*\right) \;\; \geq \;\; 0 \qquad \forall v \in V \tag{2.14}$$

Any solution (v^*, d^*) of the above variational inequality is UE flow pattern [77].

Thus, rewriting the user problem (2.13), we seek for a user equilibrium flow vectors v^*, d^* such that it solves the following problem

$$\begin{aligned} \min_{v,d} \left(\beta t(v^*)^T (v - v^*) - B\left(d^*\right)^T \left(d - d^*\right)\right) \\ s.t \\ FeC\_ED \end{aligned}$$

Given that (v^*, d^*) solves the UP above, then, following the same lines of arguments in the previous section on the analysis of the *KKT optimality conditions*, we arrive at the following results:

$$\boxed{\begin{aligned} &\sum_{a\in A} \beta t_a(v_a^*)\delta_{ar} \geq B(d_w^*) && \forall r \in R_w, \forall w \in W \\ &\sum_{a\in A} \beta t_a(v_a^*)v_a^* = \sum_{w\in W} B(d_w^*)d_w^* && \end{aligned}} \tag{2.15}$$

In fact, the following are equivalent:

1. *KKT optimality* conditions of problem (2.13)
2. Eq.(2.15) together with the *FeC_ED*

Interpretation: The first line of Eq.(2.15) confirms the Wardrop's first principle, stating that at equilibrium, no user will increase his welfare (or benefits) by unilaterally changing his route or by deciding to or not to travel. The second line is the travel cost-benefit balance equation.

Notice that the only difference between the result of *optimality conditions* of SP (Eq.(2.12)) and that of the user problem UP (Eq.(2.15)) is the absence of the term $\beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$ in Eq.(2.15). Therefore, the following are equivalent:

1. $(\bar{v},\bar{d})$ solves the *SP*
2. $\beta\left(t(\bar{v})+\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)\right)^T(v-\bar{v})-B\left(\bar{d}\right)^T\left(d-\bar{d}\right)\geq 0 \qquad \forall v\in V$

by perturbing the link cost function $\beta t_a(v_a)$ by $+\beta v_a\frac{d}{dv_a}\left(t_a(v_a)\right)\Big|_{\bar{v}_a}$ $\forall a\in A$, the optimality results in Eq.(2.15) for the user problem become exactly the same as those in Eq.(2.12). So, by charging each user of link a, a toll equal to $\beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$, for all links in the network, the users' route choice behaviour will result in the system optimal flow pattern $\bar{v}$ of system (2.1).

Therefore, by charging each user of link a, a toll equal to $\beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$, congestion cost imposed on other users by this single user is now “internalized”, and in this way, each user faces the marginal societal cost in his route choice behaviour.

In what follows, we describe the flexibility of the tolling scheme with the help of Equations (2.12) and (2.15).

Corollary 1. *Suppose $(\bar{v},\bar{d})$ is the system optimal flow pattern for the SP, then, by utilizing Eqs.(2.12) and (2.15), any toll vector θ (other than $\beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$), whose element θ_a is the toll on link a, satisfying the following set of linear conditions will also induce the system optimal flow vector $\bar{v}$ as a user equilibrium flow:*

$$\boxed{\begin{array}{ll}\sum\limits_{a\in A}\left(\beta t_a(\bar{v}_a)+\theta_a\right)\delta_{ar}\geq B(\bar{d}_w) & \forall r\in R_w,\forall w\in W\\ \sum\limits_{a\in A}\left(\beta t_a(\bar{v}_a)+\theta_a\right)\bar{v}_a=\sum\limits_{w\in W}B(\bar{d}_w)\bar{d}_w & \end{array}} \tag{2.16}$$

which we can condense in matrix form as

$$\boxed{\begin{array}{c}\Lambda^T(\beta t(\bar{v})+\theta)\geq\Gamma^T B(\bar{d})\\ (\beta t(\bar{v})+\theta)^T\bar{v}=B(\bar{d})^T\bar{d}\end{array}} \qquad EqC\_ED \tag{2.17}$$

Here, as before Λ denotes the arc-path incident matrix and Γ denotes the OD-path incident matrix for the network. The acronym EqC_ED reads *equilibrium constraint for elastic demand*. Now compare Eq.(2.16) and Eq.(2.12) and notice that we have only replaced the fixed vector $\beta\bar{v}_a\frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$ with a (variable) vector θ.

Observe from Eq.(2.17) that

$$\theta^T\bar{v} \quad = \quad B(\bar{d})^T\bar{d}-\beta t(\bar{v})^T\bar{v} \tag{2.18}$$

The *RHS* of Eq.(2.18) is a constant for all first price vectors θ. This shows that, when demand is elastic, then, the total toll revenue collected in a network is constant no matter the toll vector used [79, 78]. It further shows that though the link tolls are not unique, it is not possible that a link toll may be infinitely large as in the case of a fixed demand model (see Chapter 4 for an elaborate study on toll bounds)

Note that the marginal congestion cost pricing ($MCCP$) toll vector $\beta\bar{v}^T\nabla t(v)$ whose element $\beta\bar{v}_a\frac{d}{dv_a}(t_a(\bar{v}_a))$ is the toll for link a is valid for Eq.(2.16)

When demand is fixed, we will later see in section 3.2.1 that Eq.(2.17) becomes

$$\boxed{\begin{array}{l}\Lambda^T(\beta t(\bar{v})+\theta)\geq\Gamma^T\lambda\\ (\beta t(\bar{v})+\theta)^T\bar{v}=(\bar{d})^T\lambda\end{array}}\qquad EqC\_FD \tag{2.19}$$

where λ is a free scalar representing the minimum route travel cost for a given OD pair. The acronym EqC_FD reads *equilibrium constraint for fixed demand.*

Remark: With the aid of Equation (2.16), the following secondary objectives on the nature of the tolls can be readily defined: minimizing the total toll revenue, setting the tolls collected to a specific amount (when demand is fixed), minimizing the number of toll booths, and minimizing the maximum link toll over all links.

To summarize, we state below the algorithm that implements the first-best pricing scheme.

Algorithm for First Best Pricing scheme with Elastic Demand

1. Solve the system problem **SP** (2.1) to get the desired optimal link flow vector $\bar{v}$ and corresponding OD demand vector $\bar{d}$.
2. Find any toll vector θ satisfying EqC_ED (Eq.(2.16) or (2.17)) with secondary objectives on tolls if necessary.
3. Update the travel cost on link a to include θ_a, $\forall a\in A$.

In the foregoing, we have assumed that there is the possibility of tolling every link. As mentioned earlier, this type of tolling scheme is the so called first-best pricing. In practice, it may be infeasible to toll all road segments, to this, a pricing scheme which does not necessarily put tolls on all links has been proposed and studied in the past. It is the so called second-best pricing scheme. Below we discus this scheme.

2.1.7 Second-best toll problem formulation

In practice, it is very unlikely that all links and all users are tolled at all time periods of the day. This may be due to political reasons, poor acceptance rate, cost of implementation, etcetera. It is in view of this that the so called second-best pricing is gaining more practical grounds than the first-best counterpart. In second-best pricing, the system optimum as achieved in program (2.1) is no longer guaranteed. On the other hand, it may be that a first-best toll is feasible for the second-best scheme, then, in that case, the first-best toll is a solution for the second-best scheme. In general, we do not expect the first-best tolls to coincide with the second-best tolls due to the restriction on tolls. If such a coincidence does not occur, the system problem is solved simultaneously with the user problem in order to determine the best toll for the societal welfare.

Thus, the second-best pricing problem can be stated in words as follows:

max (the Social Welfare or Economic Benefit)
s.t
The flows and demands are feasible
The flows and demands are in (tolled) user equilibrium
Possible conditions on tolls

Algorithm for Second Best Pricing scheme with Elastic Demand

Given that Y is the set of links in the network that cannot be tolled, formally, the following describes the steps involved in implementing the second-best tolls when demand is elastic.

1. Solve the system problem **SP** ((2.1)) to obtain the optimal flow pattern $(\bar{v}, \bar{d})$.
2. Find the solution set $\mathcal{F}$ containing any social toll vector θ satisfying the equilibrium condition EqC_ED (Eq.(2.16)) and the extra conditions on tolls.
3. Check if $\mathcal{F}$ is empty, if NO, GOTO step 4 to compute the corresponding first-best social tolls, else GOTO step 5 to compute the second-best social tolls.
4. The vector θ computed in step 2 is the first-best social toll vector. Update the link cost function to $\beta t_a(v_a) + \theta_a;\ \forall a \in A$ where θ satisfies all conditions on tolls.
5. STOP.
6. By possibly using $\bar{v}, \bar{d}$ as the initial flow vectors, solve the following "bi-level" toll pricing problem:

$$\begin{aligned} \underset{v,d,\theta}{min}\, Z = C(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma \\ s.t \\ FeC\_ED\,(\text{see Eqn } 2.1) \\ EqC\_ED\,(\text{see Eqn } 2.16) \\ \theta_a = 0 \qquad \forall a \in Y \end{aligned} \tag{2.20}$$

The objective maximizes the system's welfare which is the controller's aim. The first two constraints ensure that the flow resulting from the above system is a feasible user equilibrium flow pattern. The last constraint ensures the feasibility of the toll pattern. The formulation in step could be seen as a mathematical program with equilibrium constraint. The equilibrium constraint though, has been transformed to constraints that ensure user equilibrium using the *KKT optimality* condition. Since the program above has a non-linear constraint (see Eq.(2.16) with $\bar{v}$ free), the entire problem is non-convex. The algorithm above has a unique solution in v if the system (SP) and the user (UP) problems have unique solutions. This unique solution may be difficult to achieve by non linear optimization tools though.

Note that if system problem SP (system (2.1)) and user problem UP (system (2.13)) have solutions $(\bar{v}, \bar{d})$ and (v^*, d^*) respectively and the functions *t(v) and B(d)* are continuous and monotonic or linear in v and d respectively, then, the second-best algorithm in step 1-5 above has a solution $(\tilde{v}, \tilde{d})$ with the objective value

$$\tilde{Z} = C^k(\tilde{v}) - \sum_{w \in W} \int_0^{\tilde{d}_w} B_w(\varsigma) d\varsigma$$

in the interval

$$\left[\sum_{k \in K} C^k(\bar{v}) - \sum_{w \in W} \int_0^{\bar{d}_w} B_w(\varsigma) d\varsigma \quad , \quad \sum_{k \in K} C^k(v^*) - \sum_{w \in W} \int_0^{d_w^*} B_w(\varsigma) d\varsigma \right]$$

provided the derivatives $(\nabla t(v), \nabla B(d))$ exist. The argument is very easy to see since if the algorithm terminates in step 4, then the feasible flow pattern is the same as the system optimum flow pattern (SP) and the total system cost is $\bar{Z}$ (i.e $\tilde{Z} = \bar{Z}$) which is the best one can get. On the other hand, if the algorithm does not terminate in step 4, then, the program in step 5 is solved. Note that solution to the user problem UP with objective value Z^* is a feasible solution of the program in step 5 (i.e with $\theta_a = 0$; $\forall a \in A$). Search for a better solution will force some link tolls $\theta_a \, \forall a \in A \backslash Y$ to be non zero, that is $Z^* \geq \tilde{Z}$. Therefore, $\tilde{Z}$ is bounded below by $\bar{Z}$ and above by Z^*, or

$$\bar{Z} \leq \tilde{Z} \leq Z^*$$

2.1.8 Traffic assignment and marginal congestion pricing with non-separable link travel time functions

In the above formulations, we assumed that the link travel time functions are separable, i.e., $t_a(v) = t_a(v_a)$. Such assumption is not always realistic. There are some cases where flow interactions must be considered, for example, the heavy traffic on two-way streets, the un-signalized intersections, and left-turn movements at signalized intersections. Now, we want to relax the assumption and generalize the condition to non-separable link travel time functions. In this case, the travel time function for a given link a is a function of link flows on all network links due to link flow interactions, i.e., $t_a(v) = t_a(\cdots, v_a, \cdots)$, $a \in A$. Two types of link flow interactions may arise, namely, symmetric and asymmetric link flow interactions (see [77]). For asymmetric link flow interactions, characterized by

$$\frac{\partial t_a(v)}{\partial v_b} \neq \frac{\partial t_b(v)}{\partial v_a}, \; for\, some\, a, b \in A, \; a \neq b$$

then, the variational inequality (VI) as formulated in Eq.(2.14) solves the *User Problem* with $t_a(v) = t_a(\cdots, v_a, \cdots)$, $a \in A$. The solutions to the *KKT optimality conditions* for the VI exist because the functions $t_a(v)$ are continuous and the feasible set V is compact. Further, the VI has a unique solution since $t_a(v)$ satisfies the conditions given in *Assumption 1.*

On the other hand, symmetric link flow interactions are characterized by

$$\frac{\partial t_a(v)}{\partial v_b} = \frac{\partial t_b(v)}{\partial v_a}, \forall a, b \in A, \; a \neq b$$

This means that the marginal effect on one link's travel time (say link a) inflicted by another link's flow (say link b) is equal to the marginal effect on link b's travel time inflicted by link a's flow. An equivalent mathematical program that can generate a *User Equilibrium* flow pattern can be constructed as follows (see [77]):

$$\min_{v,d} \int_0^v \beta t(u)du - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma)d\varsigma$$

The marginal congestion cost pricing (*MCCP*) toll with separable case is now given by

$$\theta_a = \sum_{b \in A} \beta v_b \frac{\partial t_b(v)}{\partial v_a}\bigg|_{v=\bar{v}} \qquad a \in A \tag{2.21}$$

where $\bar{v} = (\cdots, \bar{v}_a, \cdots)$ in Eq.(2.21) is the optimum solution to the system problem (Eq.(2.1)) with link flow interactions, i.e., $t_a(v) = t_a(\cdots, v_a, \cdots)$, $a \in A$. Eq.(2.21) is the marginal cost inflicted on all network users due to one extra user of link a. To achieve the system optimal flow $\bar{v}$, every user of link a is required to pay the amount present in Eq.(2.21).

Note that the statements of *Corollary 1* and the second-best pricing model are all valid for the non-separable link travel time functions with $t_a(v) = t_a(\cdots, v_a, \cdots)$, $a \in A$.

It is important to note that, though the convexity assumption on the link travel time functions for a separable case ensures that the *SP* program in Eq.(2.1) has a unique link flow solution $\bar{v}$, it is not the case for the non-separable link travel time function. In fact, to ensure a unique *SP* solution $\bar{v}$ in Eq.(2.1) for a special case of symmetric link flow interactions (see [77]), the following conditions must be met:

1. The travel time on each link is an increasing and convex function of the flow on that link, that is

$$\frac{\partial t_a(v)}{\partial v_a} > 0, \; \frac{\partial^2 t_a(v)}{\partial v_a^2} > 0, \; a \in A$$

2. The main dependency of a link's travel time in on its own flow.

$$\frac{\partial t_a(v)}{\partial v_a} > \sum_{b \in A, b \neq a} \frac{\partial t_a(v)}{\partial v_b}$$

For the asymmetric case, we need that the Hessian of the *SP* is positive definite which is difficult to establish. Consequently, non-separable link travel time functions may lead to a sub-optimal and/or non-unique *SP* solutions. This is in fact a draw back of using such models.

2.2 Traffic externalities of conflicting nature

2.2.1 Traffic externalities

Over the years, many countries have implemented the so called congestion pricing schemes. Following the positive results of such schemes on congestion, they have also claimed that the societal welfare in general has also been positively affected. In the literature, many spacial economists have used a function that considers only the travel time and the user's benefit to optimize the social welfare in the transportation sector. Many have over time neglected the impact of other traffic externalities such as air pollution, noise pollution, traffic safety and pavement damage among others on the social welfare. Road pricing that neglects these other externalities can lead to a bizarre network situation. In other words, a good road pricing model that maximizes social or economic welfare must involve not only the simultaneous minimization of travel time and users' dis-benefits, but also minimization of road accident, road damage, noise and air pollution and others. To capture almost all the damages a network user causes other users, the environment, residents and the future generations, we will henceforth incorporate in our models the most important traffic externalities in the definition of the social welfare.

Emissions

Air pollution is the introduction of chemicals, particulate matter, or biological materials that cause harm or discomfort to humans or other living organisms, or damage the natural environment, into the atmosphere. Though it is always difficult for all countries around the world to agree on a common project, the issue of climate change is an exceptional case. Everyone agrees that our climate must be protected. Due to technological advancement and human actions, our climate and environment deteriorate every second of the day. The negative effects of human actions on the environment as well as on humans include; global warming, ozone depletion, smog, haze, invisibility, acid rain, respiratory problems, eye irritation, restlessness and discomfort among others.

A greenhouse gas, carbon dioxide (CO_2), though vital for living organisms, is one of the major causes of global warming and the so called acid rain. It traps the heat emitted by the earth's surface thus increasing the temperature of our environment. When it reacts with atmospheric water vapour or simply water, a very weak acid called carbonic acid (*trioxocarbonate (iv) acid*) is formed. This weak acid (though unstable) can slightly increase the acidity of an unpolluted rain.

Carbon monoxide (CO) is a colourless poisonous gas which forms a stable compound with haemoglobin in the blood when inhaled by living things. This causes a reduction in the oxygen transportation from the lungs to the body cells. High concentration of CO can increase the risk of cardiovascular problems and impede the psychomotor functions.

An organic compound methane (CH_4) is one of the three main compounds that causes global warming besides CO_2 and water vapour ($H_2O\,(g)$).

Sulphur oxides (SO_x) when dissolved in atmospheric water vapour or rain cause acid rain. They also cause lung irritation.

Some nitrogen oxides (NO_x) compounds are toxic. They cause eutrophication (nutrient overload in water bodies), contribute towards the formation of smog, and are known to be ground level ozone precursors. These toxic oxides of nitrogen also cause ill health in humans and other animals, and these include: decrease in pulmonary function, inflammation of the lungs and immunological changes. Reaction of NO_2 with water droplets results in nitric acid (HNO_3) which again, causes acid rain.

Volatile organic compounds ($VOCs$) are one of the major causes of aerosols. They are very dangerous to health, and they are also known to be a precursor to the formation of ground level ozone. When NO_x reacts with *VOCs*, ozone O_3 is released. Ozone, which is beneficial in the upper atmosphere where it protects the Earth by filtering out ultraviolet radiation, has been identified as one of the leading causes of chronic respiratory diseases when found at ground level. Eye inflammation has over time been associated with ground level ozone.

Particulate matters PMs are solid particles suspended in the atmosphere. They include; re-suspended road dust, smoke, and liquid droplets. *PMs* can cause chronic and acute bronchitis, lung cancer, chest illness and chronic respiratory diseases when inhaled.

The presence of NO_x, PM_{10}, SO_2, CO, CO_2, O_3, VOC (HC), and lead (Pb) in the atmosphere has extremely been associated with road traffic;

NO_x are formed when fuel is burned at high pressure and temperature conditions. This induces the dissociation and subsequent recombination of atmospheric nitrogen (N_2) and oxygen (O_2), a reaction that generates NO_x.

PM_{10} are released into the atmosphere from so many sources. Brake pads and tires of motor vehicles are examples of such sources. Reaction of gases (e.g. NO_x, SO_2, and NH_3) from burning fuel with atmospheric water vapour leads to suspension of particulate matter in the atmosphere. Solid carbons leaving the exhaust pipes of vehicles in the form of smoke, constitute part of the solid particles seen in the atmosphere.

SO_x are one of the principal emissions from diesel engines.

CO is released into the atmosphere when fuel combustion is incomplete. Reaction between CO and atmospheric oxygen releases CO_2 into the atmosphere. In The Netherlands, it has been noted that traffic and transportation are responsible for approximately 20% of the emitted CO_2. CO_2 emission is proportional to the vehicle's fuel consumption rate, which in turn, depends on the smoothness of the traffic flow [74].

$VOCs\,(HC)$ can either be released into the atmosphere as a by-product of incomplete fuel combustion or as a vapour due to fuel evaporation.

Photochemical reactions which involve principally nitrogen oxides (NO_x), oxygen (O_2), and hydrocarbons HCs, in the presence of sunlight release (ground level) ozone (O_3) into the atmosphere.

Most fuels contain lead compounds to prevent knocking in the engine. When these fuels burn, the lead compounds are released into the atmosphere. Lead compounds when inhaled can be very injurious to health.

Noise pollution

Noise pollution is defined to be an annoying and potentially harmful environmental noise. It can be as a result of factory machines or road traffic among others. Road traffic noise results from two main factors; propulsion noise and tyre/road noise (rolling noise). Factors that influence the sound pressure level are traffic volume, speed, traffic composition (vehicle types), road design (i.e. slopes, crossings, speed bumps) and reflection, absorption and dispersion (i.e. road surface, walls, trees) [74]. The adverse effects of traffic noise include; annoyance, disturbance, high blood pressure, certain cardiovascular diseases, limited mental illness, lethal heart attack and restlessness. These effects are already recognised by the world health organisation (WHO) as serious health problems on humans. Research has proven that sounds above 55dB are potentially dangerous to health [17]. In 2000, it was recorded that more than 44% of the EU25[1] population (about 210 million people) were regularly exposed to road traffic noise level of over 55dB [17], a level, as stated above, that is dangerous to health. Research reveals that three quarters of Dutch houses experience a cumulative sound pressure of over 50dB [74]. It might be interesting to note that, the social costs of traffic noise in the EU22[2] amount to at least €40 billion per year (0.4% of total GDP). The bulk of these costs (about 90%) is caused by passenger cars and lorries [17].

Traffic safety

Road traffic accident is increasing a subject of concern in most cities around the world. It has become an urgent point of attention for the government as well as individuals since it involves immediate loss of life and severe injuries. Amazingly, it is also a major concern for insurance companies who seek for less frequent accident occurrence since this would translate to making more profit. As stated in [74], traffic safety can be discussed under three subjects; objective and subjective traffic safety, and external safety. Objective safety quantifies the traffic safety. It divides the actual number of crashes into fatal, injuries (combined with casualties) and material damages. Factors influencing objective traffic safety include; human (e.g. use of alcohol and high speed), vehicle type (e.g. mass differences), and thirdly, the road type/design. Full description of other subjects can be seen in [74].

Though the number of death and/or injury as a result of road accidents is on the increase in most countries, it may be worth mentioning that despite the increase in traffic volume, an EU country, The Netherlands (together with others) is becoming increasingly safer over the past years. The trend that describes the number of injuries per year in The Netherlands is steeply approaching ‘zero’ [74].

Road damage

Road pavements are built according to specifications. Factors that determine the type of a pavement include; geographical location and feasibility, type of vehicles that use the pavement, and financial availability among others. When heavy vehicles start using roads meant for small cars, then, it is very likely that the pavements start to dilapidate. The adverse effects of this road dilapidation range from the high cost of repair to road accidents. Annoyance and discomfort

[1]EU25 refers to EU27 except Cyprus and Malta.
[2]EU22 refers to EU27 except Cyprus, Estonia, Latvia, Lithuania and Malta.

to users are also results of dilapidated roads. When the traffic flow keeps exceeding the construction strength or the capacity of a pavement, the pavement also starts to crumble. This means that even when road pavements are used according to vehicle specifications, dense traffic can pose a serious danger to the durability of roads. Loads, which are the vehicle forces exerted on the pavements, can be characterized by tire loads, axle and tire configurations, vehicle speed, traffic distribution across the pavement and load repetition. Tire loads are the forces exerted due to tire-pavement contact. Axle and tire configuration describes how many tire contact points on a pavement, and how close they are to each other. When tires are close to each other, the pressure exerted per pavement area is increased. Slow and steady vehicles tend to create more damage to pavements. So, reducing traffic congestion will help preserve pavements to some extent. Since continuous heavy traffic on a road segment will cause this road segment to deteriorate, good traffic distribution is necessary to preserve our costly infrastructures. In this thesis, we define the pavement structural design by quantifying all expected loads a pavement will encounter over its design life. We do this by using the equivalent single axle loads ESAL which converts the wheel loads of various magnitudes and repetitions, to an equivalent number of standard (or equivalent) loads based on the amount of damage they cause to the pavement. The commonly used standard load is the 18,000lb equivalent single axle load. As a rule of thumb, the load equivalent factor LEF of each vehicle (and also the pavement/infrastructure damage imparted by each vehicle) can be roughly determined from $\left(\frac{vehicle\,weight(lb)}{18{,}000}\right)^4$ [52].
Since the man-hour is precious, and the cost of road maintenance and health care very high, it means that huge amount of money is being lost every single second from the effects of road traffic externalities. Knowing all these, we can confidently say that minimizing the mentioned effects would translate to reducing huge costs for both users and non users of the road, for the government and for organizations like insurance companies, and above all, protecting our planet Earth, and preserving it for yet unborn generations. To achieve this aim, we must put into consideration all the mentioned effects and search for a way to minimize the cost generated by each effect. A closer look tells one that we are already confronted with a multi-objective problem, and this leads to the *introduction of a multi-objective problem* in the next subsection.

2.2.2 Multi-objective optimization and its meaning

Most real life optimization problems require the simultaneous optimization of more than one objective function. This is because many of these problems are defined in many objectives. In most cases, these objectives are in conflict with each other and may or may not be equally important. Examples of realistic optimization problems that involve more than one objective function include the following:

1. In the bridge construction, a good design is characterized by low total mass and high stiffness.
2. Most aircraft designs require simultaneous optimization of fuel efficiency, payload and weight.

3. A good road pricing model that maximizes social or economic welfare must involve the simultaneous minimization of travel time, noise and air pollution, road accident, road damage and maximization of user benefit.
4. An acceptable/fair road pricing scheme must include (usually) conflicting interests of various stakeholders and the road users.
5. The traditional portfolio optimization problem attempts to simultaneously minimize the risk and maximize the fiscal return.
6. In chemical design, or in the design of a ground water bioremediation facility, objectives to look out for include total investment and net operating costs.
7. In car manufacturing industries, two objectives; minimization of the noise a driver hears and maximization of ventilation are used to define a good sun roof design.
8. In logistics and supply chain, a good scheduling of truck routing minimizes dead head miles while equalizing the work load among drivers.
9. An optimization problem in radiation therapy planning always aims at minimizing radiation dose on the normal tissues while maximizing dose on the tumour regions.

In these and many other cases, it is unlikely that the different objectives would be optimized by the same alternative parameter choice. This means that some trade-offs are needed between criteria to ensure a satisfactory model. Solving a multi-objective problem often results in a multitude of solutions, and not all these solutions are of interest [36]. For a solution to be interesting, there exist a dominance relation between the solution considered and the other solutions. We say that a solution vector $\bar{v}$ dominates another solution vector v^* if:

$\bar{v}$ is at least as good as v^* for all objectives, and

$\bar{v}$ is strictly better than v^* for at least one objective.

In general, a multi-objective problem has no optimal solution that could optimize all objectives simultaneously since these objectives are conflicting. But there exists a set of non-dominant or non-inferior or equally efficient alternative solutions, known as the Pareto optimal set. A Pareto optimal solution has the property that it is not possible to reduce any of the objective functions without increasing at least one of the other objective functions.

Definition

If for objective $k \in K$, $C^k(v)$ denotes the cost or objective function (to be minimized), then a solution vector $\bar{v} \in V$ dominates a solution vector $v \in V$ if and only if the following holds:

$$\begin{aligned} C^k(\bar{v}) &\leq C^k(v) \qquad \forall k \in K \quad \text{and} \\ C^j(\bar{v}) &< C^j(v) \qquad \text{for at least one } j \in K \end{aligned}$$

the solution $\bar{v} \in V$ is *Pareto optimal* if there does not exist any other solution vector $v \in V$ that dominates $\bar{v}$.

The line that connects the set of all Pareto points (sometimes called efficient points) to a multi-objective optimization problem is called the Pareto or efficient frontier [34]. Next we turn our attention to the multi-objective theory of road pricing and the mentioned conflicting externalities.

2.2.3 Traffic externalities and multi-objective optimization of road pricing

We state again that road tolls/pricing is a well accepted technique in transportation economics to combat traffic externalities such as congestion, emission, noise, safety issues, etcetera. The road pricing problem is to determine how much toll to place and on which link such that traffic is efficiently distributed in a given network. Efficiency is used here to mean traffic pattern that optimizes the externalities of interest. Due to the conflicting nature of these externalities, a road pricing scheme that checks congestion, for example, may hugely increase the emission of gases into the atmosphere, and allow vehicles to use the urban/inland (shorter) road which may endanger the safety of residents in the form of accidents and inhalation of poisonous exhaust gases, and even further dilapidates the pavements. On the other hand, a road pricing scheme that promotes safety may lead to "slow" moving vehicles or assignment of all vehicles to the safest route, and this of course will result in a heavily congested route which in turn leads to huge man-hour loss in congestion, and again, pavement dilapidation. Furthermore, a pricing scheme that reduces noise will channel traffic to non residential roads, and this may lead to under utilization of urban/inland roads while other roads are heavily loaded with vehicles. Recall that road dilapidation is enhanced by congestion, so heavy loaded roads decay faster. The story of the conflicting nature of the traffic externalities is that when you try to minimize the cost of one externality, then the cost of two or more others is on the increase.

As we mentioned earlier, there does not exist a flow vector v that absolutely optimizes all objectives simultaneously, otherwise, there is no need to solve the problem with multiple objectives. Our aim in this thesis, therefore, is to find a satisfactory and acceptable trade off between the objectives. Starting with the first-best pricing scheme, we will extend our results to the second-best pricing scheme. Let us first look at the objectives and how the 'optimal' solutions can be obtained, and later on, we will show how these 'optimal' solutions can be achieved on road networks using tolls.

Here we have a look at some concepts from multi-objective optimization. One possible solution concept is to compute some Pareto points and see how these points are reflected in objective values.

Define v^{*k} to be the optimal flow vector of objective k. This means that $C^k(v^{*k})$ is the optimal value (in the absolute sense) of objective k. We will call v^{*k} the ideal vector point of objective k, and $C^k(v^{*k})$, the ideal objective value of objective k, and C^*, the ideal objective vector containing all ideal objective values. Observe that for any objective $k \in K$, we have that $v^{*k} \neq v^{*j}$ for at least one objective $j \in K$, where $k \neq j$, otherwise, the objectives would not be in conflict with one another. $C^k(v^{*k})$ will serve as the lower bound for objective k. Since a multitude of feasible Pareto optimal solutions is obtained in a multi-objective optimization problem, the question is, which of these solutions will be chosen? This decision depends solely on the choice of the decision maker (dm). As some objectives may be more important to the dm than others, the solutions more favourable to less important objectives are discarded and solutions favourable to more important objectives are selected for further analysis. The judgement of which solution to choose among the remaining or equally important objectives is always a challenging task. This is because, in Pareto sense, no solution is better than the other. It may be possible for the decision maker to define a utility or value function to enable him evaluate or quantify the trade-offs or the objective values [36]. If an explicit utility function can be constructed, then, the objectives can be aggregated into one criterion, and in this way, the multi-objective problem reduces to a single objective problem. The definition of a good value or utility function is very difficult in practice though. No matter the valuation of the *dm*, for efficiency/optimality reasons, the final accepted solution should be a point on the Pareto frontier.

Let us first assume that we have only one decision maker, say the central government. Without loss of generality, we assume that the *dm* wants to keep as low as possible the costs of total system travel time, emission, noise, safety, and road damage. Decrease in travel time as we stated earlier encourages the use of short inland roads at high speeds, but these translate not only to high emission and noise in the urban areas but also endanger the safety of inhabitants and the motorists. Since these objectives are conflicting, it means that it is not possible to find a feasible flow vector v^* that results in ideal objective value for all the objectives simultaneously. We will then seek a feasible flow pattern (or vector) $\bar{v}$ that is as close as possible to the interest of the *dm*. Since all our objectives are monetized, and for the fact that the *dm*'s sole interest is to minimize the entire network cost, give no objective function preference over the other. Since the efficient solution points for the multi-objective problem are all on Pareto frontier, any choice for one solution point that favours one objective may lead to increase in the cost of another objective value. This move may or may not decrease the entire network cost. We then seek a flow vector $\bar{v}$ on the Pareto frontier that minimizes the total network cost.

The *dm*'s objective to keep the gap (with respect to the ideal value) as small as possible for each $k \in K$, and his intention to minimize the total trade off translate to one aim; minimizing the entire network costs. This leads to the following objective formulation:

$$\begin{aligned} \min_{v^k} \ \sum_{k \in K} w^k \left(C^k(v^k) - C^k(v^{*k}) \right) & \\ s.t & \\ v \in V & \end{aligned} \tag{2.22}$$

where $w^k \in \mathbb{R}$ is the weight of the objective k and $C^k(v^{*k})$ is the ideal objective value of objective k. For our problem, $w^k = \omega \quad \forall k \in K$ assuming that we scaled the various costs according to their monetary value.

The objective tends to minimize the gap between objective value and the ideal objective value for all objectives. By the summation, system (2.22) minimizes the sum of the trade off, and thus ensuring that the trade-offs made between objectives result in a minimum system cost. Another look at (2.22) reveals that the formulation is simply minimizing the entire network costs. Since all dominant solutions lie on the Pareto frontier. The above problem is Pareto optimal. The proof is similar to the ones given in [36].

System (2.22) translates to a weighted sum method of multi-objective optimization. The weighted methods have a shortcoming of not being able to find all the Pareto optimal solutions of non-convex problems. The objective functions studied in this thesis are either convex or linear in v, thus, the feasible objective space is convex.

Remark

An important feature of a multi-objective problem is the connectedness of the sets of Pareto optimal and weakly Pareto optimal solutions. It is often useful to know how well one can move continuously from one Pareto optimal solution to another. Steuer [64] proves that the Pareto optimal set of a multi-objective optimization is connected when the objectives are linear. Warburton [71] proves the connectedness of Pareto optimal set for a convex case.

2.2.4 Single-leader multi-objective road pricing model

If we assume that a single leader or a stakeholder sets toll on the transportation network with the aim of minimizing the cost arising from the aforementioned externalities in the form of multiple objectives, then as stated earlier, this decision maker is faced with an option of choosing a point on the Pareto frontier. Assuming we monetize all externalities, then, for equity reasons (equal weights on all objectives), the redefined multi-objective version of system Eq.(2.1) to keep the total system monetary cost as low as possible is given as

$$(MO): \quad \underset{v,d}{min}\ Z = \sum_{k\in K} C^k(v) - \sum_{w\in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$\begin{array}{rcll} & s.t & & \\ v & = & \Lambda f & [\psi] \\ \Gamma f & = & d & [\lambda] \\ f & \geq & 0 & [\rho] \\ d & \geq & 0 & [\vartheta] \\ g(v) & \leq & 0 & [\xi] \end{array} \tag{2.23}$$

The user problem remains as given in subsection 2.1.6. Let $\bar{v}$ be the solution of *MO,* then the results in subsections 2.1.6 and 2.1.7 are still valid with the marginal congestion cost pricing $(MCCP)$ $\beta\bar{v}_a \frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$ for link a as given in Eq.(2.12) being replaced by

$$Q_a = \left[\beta v_a \frac{d}{dv_a}\left(t_a(\bar{v}_a)\right) + \alpha_a \frac{d}{dv_a}\left(e_a(\bar{v}_a)\right) + \gamma \frac{d}{dv_a}\left(n_a(\bar{v}_a)\right) + \frac{d}{dv_a}\left(i_a(\bar{v}_a)\right) + \varrho \frac{d}{dv_a}\left(s_a(\bar{v}_a)\right)\right] \tag{2.24}$$

Here $t_a(v_a)$ is the travel time cost function on link a, $e_a(v_a)$ is the emission cost function on link a, $n_a(v_a)$ is the noise cost function on link a, $i_a(v_a)$ is the infrastructure-damage cost function on link a, and $s_a(v_a)$ is the safety cost function on link a. Further, β is the monetary value of time per minute (VOT), α_a is the monetary value of emission per gramme for link a which depends on the urbanization (among other factors), γ is the monetary equivalent of *1dB*(A)

defined for a certain noise level, and ϱ is the 'average' monetary cost of an injury crash.

For the first-best pricing scheme, instead of charging the *MCCP* term $\beta \bar{v}_a \frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$ on link a, the term Q_a in Eq.(2.24) is charged. We call term Q_a the marginal social cost pricing (*MSCP*) term. The terms in Eq.(2.24) is interpreted as follows:

$\beta v_a \frac{d}{dv_a}\left(t_a(\bar{v}_a)\right)$ is the additional travel cost imposed on all existing users of link a by an additional user of this link.

$\alpha_a \frac{d}{dv_a}\left(e_a(\bar{v}_a)\right)$ is the additional cost to the environment due to emission caused by an additional user of link a.

$\gamma \frac{d}{dv_a}\left(n_a(\bar{v}_a)\right)$ is the additional cost on the society due to increase in noise level caused by an additional user of link a.

$\frac{d}{dv_a}\left(i_a(\bar{v}_a)\right)$ is the cost for the road damaged caused by a single car using link a.

$\varrho \frac{d}{dv_a}\left(s_a(\bar{v}_a)\right)$ is the cost for the increase in risk level on link a due to an additional user of this link.

Thus, by charging each user of link a, a toll equal to Q_a, all the costs imposed on other users, on the environment, and on the society by a single user may now be internalized. In this way therefore, each user faces the marginal societal cost in his route choice behaviour. Such behaviour, we claim, leads to optimal use of the system.

It is worthwhile noting that the one-objective road pricing, or specifically marginal congestion cost pricing *MCCP*, which considers only travel time and compels the users towards the desired flow, is a special case of the model above.

As discussed in section 2.1.8, if we now assume a non-separable link cost functions such that there are link flow interactions, then the generalized form of Eq.(2.24) is given as:

$$Q_a = \left[\sum_{b\in A}\beta v_b \frac{\partial t_b(\bar{v})}{\partial v_a} + \sum_{b\in A}\alpha_a \frac{\partial e_b(\bar{v})}{\partial v_a} + \sum_{b\in A}\gamma \frac{\partial n_b(\bar{v})}{\partial v_a} + \sum_{b\in A}\frac{\partial i_b(\bar{v})}{\partial v_a} + \sum_{b\in A}\varrho \frac{\partial s_b(\bar{v})}{\partial v_a}\right] \tag{2.25}$$

where $\bar{v} = (\cdots, \bar{v}_a, \cdots)$ in Eq.(2.25) is the optimum solution to the system problem (Eq.(2.23)) with link flow interactions, i.e., $t_a(v) = t_a(\cdots, v_a, \cdots)$, $a \in A$. Hence, Q_a is the total traffic externality cost inflicted on the entire network by a single user of link a.

2.3 Conflicting interests of stakeholders

Even with its rich potentials of alleviating most of our traffic worries, road pricing has suffered setbacks when it comes to practical implementation of the scheme. One major setback is suffered due to political reasons - where involving parties and stakeholders who decide on road pricing schemes have different objectives (mostly conflicting) that they want the proposed scheme to offer. The other main setback is due to poor public acceptance of the scheme - road users always perceive themselves as passive instead of active players when debates on road pricing or decisions take place. Specifically, for example, insurance companies would solicit for a toll pattern that minimizes road accidents and have little or no interest in congestion, whereas the Ministry of Economics may be interested in minimizing man-hour loss in the traffic to boost productivity. On the other hand, environmentalists would lobby for a pricing scheme that limits traffic emissions while showing little interest on other externalities. Still more, one region within a country may set tolls to optimally distribute traffic on her regional network irrespective of the flow pattern and/or the tolling scheme of other regions. These optimal regional tolls may unfavourably affect traffic flows in both near and far away regions. We will see later in Chapter 3 that our stakeholders' model can easily be transformed into a model involving regions of different interest.

2.4 Game theoretical approach

Following subsection 2.3.1, it is now clear that every stakeholder and all regions would want their voice be heard during the toll setting. Another look tells one that we are facing a natural game where each player has an objective which may be in conflict with other players' objective. A player's aim would be to push/lobby as hard as he can to achieve the best for himself. In such a model where more than one stakeholder or decision maker controls the affairs of road pricing, then the concept of Nash equilibrium [40] from economics presents a suitable model. In the Nash equilibrium game, players play in turns to improve their individual utilities (while obeying the rules of the game) until none of the players can improve his current outcome. Due to the fascinating nature of the road pricing game, I am almost being tempted to start describing the whole game procedure here, see you in Chapter 3 for the full description of the game!

2.5 Summary and conclusion

In this chapter, we have given a detailed review of road pricing, its implementations and setbacks. We described and mathematically formulated two main road pricing schemes, namely, first-best and second-best pricing schemes. We further derived a flexible tolling system for the two schemes. Major traffic externalities were introduced in this chapter together with their effects on humans and a description of their conflicting nature when represented in terms of objectives. The notion of multi-objective optimization was introduced, and few solution methods were described. The chapter concludes by classifying our multi-objective problem to a group of games we call *multi-objective games and proposed a Nash equilibrium solution approach, an approach that had not been considered before in road pricing.*

Chapter 3

Bi-level multi-stakeholder multi-objective optimization model

In Chapter 2, we discussed a one leader road pricing problem where the leader concentrates on just one objective, or group of objectives described in one utility function. In subsection 2.2.4 of that chapter, we described a model that extends the single objective road pricing model to models of multiple objectives handled by a single leader or actor. Such models have their shortcomings; when one decision maker (*dm*), (e.g. the government or a private road owner) controls the traffic flow of a transportation system through road pricing, then it is likely that some other stakeholders affected by activities of transportation may not be happy with the decisions made by this *dm*. This is because when the *dm* models the multi-objective (MO) road pricing problem, all traffic externalities are simultaneously considered with or without preference for any externality (see MO Eq.(2.23)). When preference is given, say, to congestion, then the effect of the preferred externality subdues the effect of other externalities, and this may translate to huge costs for some objectives (or stakeholders). For example, lower travel time (say high speeds) may translate to more accidents (costs for insurance companies). Even without preference to any externality, it is intuitive that stakeholders still will prefer to partake in toll setting to protect their interests. The main problem of a classical approach from multi-objective optimization is the following: supposing that each stakeholder can influence the toll setting, why should an independent player accept a situation which he can improve (his objective) by changing the tolls?

In such a situation, the classical concept of Nash equilibrium in game theory presents an appropriate alternative model. Such models are accepted in economics in situations where independent players may influence the market with their strategies in order to optimize their specific objective.

The question we would like to address from a game theoretical/economic point of view is: What happens when each stakeholder optimizes his objective by tolling the same network, given that other stakeholders are doing the same? Formally, we introduce the mathematical and economic theory behind the model.

3.1 Mathematical and economic theory

The mathematical program with equilibrium condition (Eq.(2.20)) described in Chapter 2 is a Stackelberg game where a leader or *dm* (at the upper level) moves first, followed by sequential moves of other players (road users). If we assume that various stakeholders are allowed to propose a toll (or at least influence the tolls) for a network, then users are influenced not only by just one leader as in

Stackelberg games, but by the actions of more than one decision maker. In a multi-leader-multi-follower game, the leaders take decisions (search for toll vectors θ^k, $k \in K$, that optimize their individual objectives) at the upper level which influence the followers (users) at the lower level. The followers then react accordingly (user/Wardrop's equilibrium), which in turn may cause the leaders to update their individual decisions leading to lower level players reactions again. These updates continue until a stable situation is reached. A stable state is reached if no stakeholder can improve his objective by unilaterally changing his proposed toll. Note, however, that given the stable state decision tolls of leaders, the lower level stable situation is given by the (unique) Wardrop's equilibrium. So the bi-level game can be seen as a single (upper) level game with additional equilibrium conditions (for the lower level - see Eq.(2.20)).

In the above non-cooperative scenario, each actor solves a *program with equilibrium conditions*, which is influenced by other actors' *programs with equilibrium conditions*, and this translates to an *equilibrium problem subject to an equilibrium condition.* Since a stable state upper level tolls will lead to a (unique) Wardrop's equilibrium in the lower level, our aim, therefore, is to find a Nash toll vector for the leaders (see figure 3.1).

After settling on a Nash toll vector, users represented in the upper level may search for an alternative but lower toll vector using Equation (2.17) or (2.19). Notice from figure 3.1 that we have represented the interests of road user in the upper and the same level as other stakeholders, thus making them active players in the toll setting game. This type of formulation, we hope, will go a long way to promote public acceptance of road pricing since they (users) now have a stakeholder on the "*discussion table*".

Remark: The theory described above does not necessarily mean that stakeholders have different toll collecting machines on the links, no, our model describes the Nash toll vector that can be reached during policy making or debate when stakeholders or autonomous regions/states choose not to form a grand coalition.

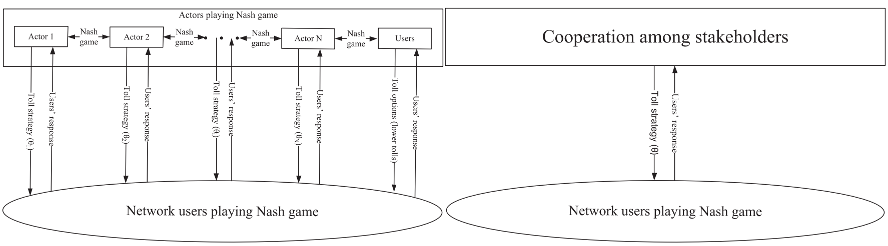


Figure 3.1: Multi-leader-Multi-follower Nash/Cooperative game model

3.2 Multi-stakeholder multi-objective road pricing model

We devote this subsection to the derivation of the game models we will use in the remainder of this thesis. In the model, we will mathematically describe the interaction between actors in the upper level and the interaction between road users in the lower level. Further, we will model the interaction between upper and lower level players in the form of a game.

3.2.1 Derivation of the pricing model for one stakeholder game

In subsection 2.1.6, we derived first-best and second-best flexible pricing schemes for elastic demand models. In this subsection, we will turn our attention to fixed demand models. In the fixed demand model, it is assumed that demands as read from the input origin-destination (OD) matrix are fixed during the modelling period. We will describe mathematically what each stakeholder does in the upper level game between the stakeholders, and what each road user does in the lower level game among the road users. The models are very much similar to the ones described in the subsection 2.1.6. With a knowledge of the derivation in subsection 2.1.6, the reader can go straight to section 3.2.2.

Stakeholders' model (upper-level)

Given that we have $|K|$ stakeholders, then a stakeholder $k \in K$ does nothing than optimizing his objective $C^k(v^k)$ which depends on the traffic flow vector v (see *Assumption 1* in subsection 2.1.6). The Eq.(2.1) equivalent for the fixed demand model for player $k \in K$ is

$$(SP_k): \qquad \min_{v^k} Z^k = C^k(v^k)$$
$$\text{s.t}$$
$$\begin{aligned} v^k &= \Lambda f^k & & \left[\psi^k\right] \\ \Gamma f^k &= \bar{d} & & \left[\lambda^k\right] \\ f^k &\geq 0 & & \left[\rho^k\right] \end{aligned} \tag{3.1}$$

The constraints are the flow feasibility conditions as described in Eq.(2.1). The bar "$^-$" on the demand d signifies that d is fixed.

As in subsection 2.1.6, $\left(\psi^k, \lambda^k, \rho^k\right)$ are the *KKT multipliers* associated with the constraints. We will refer to constraints of system (3.1) as the **Feasibility Conditions for Fixed Demand** denoted by *FeC_FD*.

Recall *Assumption 1* in subsection 2.1.6:

- We assume throughout that the link cost (or travel time) function vector $t(v)$ is continuous and satisfies $(t(v) - t(\bar{v}))^T (v - \bar{v}) > 0 \ \forall v \neq \bar{v},\ v, \bar{v} \in V$ and all functions $C^k(v)$ are continuous, strictly convex, and strictly monotone (in the sense that $\partial C^k(v)/\partial v_a \geq 0 \ \forall k, a$), and the side constraints $g(v) \leq 0$ (see Eq.(2.1)), if used, are linear.

If *Assumption 1* holds for system (3.1), we now analyse the *KKT optimality conditions* of system (3.1) assuming a separable cost functions. If we let L be the Lagrangian, and $\bar{v}^k$ be the solution of the above program, then, there exists $\left(\psi^k, \lambda^k, \rho^k\right)$ such that the following *KKT conditions* hold:

$$
\begin{aligned}
L &= \sum_{a\in A} C_a^k(v_a^k) + (\Lambda f^k - v^k)^T\psi^k + (\bar{d} - \Gamma f^k)^T\lambda^k - \left(f^k\right)^T \rho^k \\
\frac{\partial}{\partial v_a^k} L &= \frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right) - \psi_a^k = 0 \quad \forall a \in A && (3.2)\\
\frac{\partial}{\partial f^k} L &= \Lambda^T\psi^k - \Gamma^T\lambda^k - \rho^k = 0 \\
&\Rightarrow \sum_{a\in A} \psi_a^k \delta_{ar} - \lambda_w^k - \rho_r^k = 0 \qquad \forall r \,\epsilon\, R_w,\, w \in W && (3.3)\\
\left(f^k\right)^T \rho^k &= 0 \qquad \forall w \in W && (3.4)\\
&\Rightarrow f_r^k \rho_r^k = 0 \qquad \forall r \in R \\
\rho^k &\geq 0 \quad OR \quad \rho_r^k \geq 0 \qquad \forall r \in R_w,\, w \in W
\end{aligned}
$$

Where $\frac{\partial}{\partial x}$ is the partial derivative with respect to x, and $\Lambda = \delta_{ar}$ is a binary parameter that equals *1* if link a belongs to a path r and *0* otherwise. We have used v^k to denote the link flow vector for stakeholder $k \in K$ with elements v_a^k representing the link flows. Eq.(3.4) is the complementarity condition.

Substituting Eq.(3.2) into Eq.(3.3) yields

$$
\sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\delta_{ar} = \lambda_w^k + \rho_r^k \qquad \forall r \in R_w,\, w \in W \tag{3.5}
$$

since $\rho_r^k \geq 0$, $\forall r \in R_w$, $w \in W$, (3.5) reduces to

$$
\boxed{\sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\delta_{ar} \geq \lambda_w^k \qquad \forall r \in R_w, \forall w \in W} \tag{3.6}
$$

As a result of the flow conservation constraint in Eq.(3.1), we derive the following:

From Eqn 3.1 we have $\quad v^k = \Lambda f^k$

$$
\begin{aligned}
&\Rightarrow \bar{v}_a^k = \sum_{w\in W}\sum_{r\in R_w} f_r^k \delta_{ar} \qquad \forall a \,\epsilon\, A \\
\sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\bar{v}_a^k &= \sum_{a\in A}\sum_{w\in W}\sum_{r\in R_w}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right) f_r^k \delta_{ar} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r^k \sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\delta_{ar} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r^k(\lambda_w^k + \rho_r^k), \qquad \text{using Eqn 3.5} \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r^k \lambda_w^k + \sum_{w\in W}\sum_{r\in R_w} f_r^k \rho_r^k \\
&= \sum_{w\in W}\sum_{r\in R_w} f_r^k \lambda_w^k = \sum_{w\in W}\lambda_w^k \sum_{r\in R_w} f_r^k, \qquad \text{using Eqn 3.4} \\
&= \sum_{w\in W}\lambda_w^k \bar{d}_w, \qquad \text{using Eqn 3.1}
\end{aligned}
$$

$$
\therefore \sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\bar{v}_a^k = \sum_{w\in W}\lambda_w^k \bar{d}_w \tag{3.7}
$$

To summarize, a feasible flow vector $\bar{v}^k$ is a (unique) solution of SP_k if and only if with a vector λ^k the following holds:

$$\boxed{\begin{aligned}&\sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\delta_{ar}\geq\lambda_w^k \qquad\qquad \forall r\in R_w,\ w\in W\\ &\sum_{a\in A}\left(\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)\right)\bar{v}_a^k=\sum_{w\in W}\lambda_w^k\bar{d}_w\end{aligned}} \tag{3.8}$$

Where $\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)$ is the total cost incurred by actor k on link a due to a single user on this link. Note that for a specific case where $C^k(v^k)=\beta v^T t(v)$, the total system travel time cost, then Eq.(3.8) becomes

$$\boxed{\begin{aligned}&\sum_{a\in A}\left(\beta t_a(\bar{v}_a^k)+\beta\bar{v}_a\frac{d}{dv_a^k}\left(t_a(\bar{v}_a)\right)\right)\delta_{ar}\geq\lambda_w^k \qquad\qquad \forall r\in R_w,\ w\in W\\ &\sum_{a\in A}\left(\beta t_a(\bar{v}_a^k)+\beta\bar{v}_k^a\frac{d}{dv_a^k}\left(t_a(\bar{v}_a^k)\right)\right)\bar{v}_a^k=\sum_{w\in W}\lambda_w^k\bar{d}_w\end{aligned}} \tag{3.9}$$

Observe from Eq.(3.9) that the cost incurred by actor k on link a is in two parts: the travel cost of a single user on link a, $\beta t_a(\bar{v}_a^k)$, and $\frac{d}{dv_a^k}\left(C_a^k(\bar{v}_a^k)\right)$- the travel cost imposed on all other users of link a by a single user of link a. Eq.(3.9) is the fixed demand version of Eq.(2.12).

Road users' model (lower-level)

As formulated in subsection 2.1.6, the user equilibrium problem for fixed demand is given by the following program: Find $v^*\in V$ such that v^* is a solution of

$$\begin{aligned}\min_v &\left(\beta t(v^*)^T v\right)\\ &s.t\\ v &= \Lambda f \qquad \psi\\ \Gamma f &= \bar{d} \qquad \lambda\\ f &\geq 0 \qquad \rho\end{aligned} \tag{3.10}$$

(ψ,λ,ρ) are the *KKT multipliers* associated with the constraints.

Again, let *Assumption 1* hold for (3.10), then, we analyse the *KKT optimality conditions* for system (3.10). If we let L be the Lagrangian, and v^* be the solution of the above program, then, there exists (ψ,λ,ρ) such that the following *KKT conditions* hold:

$$
\begin{aligned}
L &= \sum_{a\in A}\beta t_a(v_a^*)^T v_a + (\Lambda f - v)^T\psi + (\bar{d} - \Gamma f)^T\lambda - f^T\rho \\
\frac{\partial}{\partial v_a}L &= \beta t_a(v_a^*) - \psi_a = 0 \quad \forall a \in A & (3.11)\\
\frac{\partial}{\partial f}L &= \Lambda^T\psi - \Gamma^T\lambda - \rho = 0 \\
&\Rightarrow \sum_{a\in A}\psi_a\delta_{ar} - \lambda_w - \rho_r = 0 \qquad \forall r \,\epsilon\, R_w,\, w \in W & (3.12)\\
f^T\rho &= 0 \qquad \forall w \in W & (3.13)\\
&\Rightarrow f_r\rho_r = 0 \qquad \forall r \in R \\
\rho &\geq 0 \quad OR \quad \rho_r \geq 0 \qquad \forall r \in R_w,\, w \in W
\end{aligned}
$$

All notations remain as previously defined.

Substituting Eq.(3.11) into Eq.(3.12) yields

$$
\sum_{a\in A}\left(\beta t_a(v_a^*)\right)\delta_{ar} = \lambda_w + \rho_r \qquad \forall r \in R_w,\, w \in W \tag{3.14}
$$

Given that a path $r \in R_w$ has a positive flow, then Eq.(3.14) reduces to

$$
\sum_{a\in A}\left(\beta t_a(v_a^*)\right)\delta_{ar} = \lambda_w \qquad \forall f_r > 0,\, r \in R_w,\, w \in W \tag{3.15}
$$

due to the complementarity condition in Eq.(3.13).

Interpretation: Eq.(3.15) states that at equilibrium, the travel costs on all used paths $r \in R_w$ in a traffic network for a given origin-destination pair $w \in W$ are the same and equal to the parameter λ_w.

Furthermore, since $\rho_r \geq 0$, $\forall r \in R_w$, $w \in W$, from Eq.(3.14), the following holds in general

$$
\boxed{\sum_{a\in A}\left(\beta t_a(v_a^*)\right)\delta_{ar} \geq \lambda_w \qquad \forall r \in R_w, \forall w \in W} \tag{3.16}
$$

Interpretation: From Eq.(3.15) and (3.16) we thus conclude that at equilibrium, the travel costs on all used paths for a given OD pair are the same and less or equal to the parameter λ_w. This is the so called Wardrop's first principle, thus, any flow vector v^* that solves system (3.10) is indeed a user equilibrium flow vector.

From Eqn 3.10 we have

$$
\begin{aligned}
v &= \Lambda f \\
\Rightarrow v_a^* &= \sum_{w \in W} \sum_{r \in R_w} f_r \delta_{ar} \qquad \forall a \,\epsilon\, A \\
\sum_{a \,\epsilon\, A} \left(\beta t_a(v_a^*)\right) v_a^* &= \sum_{a \,\epsilon\, A} \sum_{w \,\epsilon W} \sum_{r \in R_w} \left(\beta t_a(v_a^*)\right) f_r \delta_{ar} \\
&= \sum_{w \in W} \sum_{r \in R_w} f_r \sum_{a \in A} \left(\beta t_a(v_a^*)\right) \delta_{ar} \\
&= \sum_{w \in W} \sum_{r \in R_w} f_r (\lambda_w + \rho_r), \quad \text{using Eqn 3.14} \\
&= \sum_{w \in W} \sum_{r \in R_w} f_r \lambda_w + \sum_{w \in W} \sum_{r \in R_w} f_r \rho_r \\
&= \sum_{w \in W} \sum_{r \in R_w} f_r \lambda_w = \sum_{w \in W} \lambda_w \sum_{r \in R_w} f_r, \quad \text{using Eqn 3.13} \\
&= \sum_{w \in W} \lambda_w \bar{d}_w, \quad \text{using the second constraint of 3.10} \\
\therefore \sum_{a \in A} \left(\beta t_a(v_a^*)\right) v_a^* &= \sum_{w \in W} \lambda_w \bar{d}_w \qquad (3.17)
\end{aligned}
$$

To summarize, a vector $v^* \in V$ is a solution of (3.10) if and only if with a vector λ^k the following holds:

$$
\boxed{
\begin{aligned}
&\sum_{a \in A} \left(\beta t_a(v_a^*)\right) \delta_{ar} \geq \lambda_w \qquad \forall r \in R_w,\, w \in W \\
&\sum_{a \in A} \left(\beta t_a(v_a^*)\right) v_a^* = \sum_{w \in W} \lambda_w \bar{d}_w
\end{aligned}
} \qquad (3.18)
$$

Once again, Eq.(3.18) is an equilibrium condition with v^* the equilibrated flow pattern.

Comparing Eqs.(3.8) and (3.18) reveals that each stakeholder $k \in K$ would want to achieve his *ideal* flow link flow vector $\bar{v}^k$ by setting a link toll θ_a^k given by

$$
\begin{aligned}
\beta t_a(\bar{v}_a^k) + \theta_a^k &= \frac{d}{dv_a^k} \left(C_a^k(\bar{v}_a^k)\right) \\
\textit{or} \;\; \theta_a^k &= \frac{d}{dv_a^k} \left(C_a^k(\bar{v}_a^k)\right) - \beta t_a(\bar{v}_a^k) \qquad (3.19)
\end{aligned}
$$

Note that the link toll in Eq.(3.19) can be negative, meaning that stakeholders may have to give sort of incentives or utilities to road users in order to achieve the flow vector $\bar{v}^k$. In most cases, giving out utilities in the form of money or incentives is more or less, practically infeasible. In the next chapter, we will show that it is still possible to achieve the flow vector $\bar{v}^k$ with a positive toll vector θ^k, in fact, it turned out that there exists an infinite number of positive toll vector θ^k that can be used to achieve the flow vector v^k for stakeholder k. Thus we state the following corollary:

Corollary 2. *Suppose* $\bar{v}^k$ *is the system optimal flow pattern for stakeholder* k *in the* stakeholder's model *(Eq.(3.1)), then, by utilizing Eqs.(3.8) and (3.18), any toll vector* θ^k *(other than* $\frac{d}{dv^k}\left(C^k(\bar{v}^k)\right) - \beta t(\bar{v}^k)$*), whose element* θ^k_a *is actor* k'*s toll on link* a*, satisfying the following set of linear conditions will also induce the system optimal flow vector* $\bar{v}^k$ *as a user equilibrium flow vector:*

$$\boxed{\begin{array}{ll} \sum\limits_{a\in A}\left(\beta t_a(\bar{v}^k_a)+\theta^k_a\right)\delta_{ar} \geq \lambda^k_w & \forall r \,\epsilon\, R_w, \forall w \,\epsilon W \\ \sum\limits_{a\in A}\left(\beta t_a(\bar{v}^k_a)+\theta^k_a\right)\bar{v}^k_a = \sum\limits_{w\in W}\lambda^k_w \bar{d}_w & \end{array}} \tag{3.20}$$

which we can condense in matrix form as

$$\boxed{\begin{array}{l} \Lambda^T\left(\beta t(\bar{v}^k)+\theta^k\right) \geq \Gamma^T\lambda^k \\ \left(\beta t(\bar{v}^k)+\theta^k\right)^T \bar{v}^k = (\bar{d})^T\lambda^k \end{array}} \tag{3.21}$$

Where Λ denotes the arc-path incident matrix and Γ denotes the OD-path incident matrix for the network. λ is a free variable representing the minimum route travel cost vector for the ODs.

Proof: The proof follows from the *KKT optimality condition* analysis in subsection 3.2.1. ■

Henceforth, we will call any toll vector θ^k satisfying Eq.(3.20) *a first-best toll vector for actor* k*.*

3.2.2 Mathematical model for the multi-stakeholder bi-level Nash equilibrium game

We now introduce mathematically the toll pricing game and the concept of Nash equilibrium (NE) Nash [40], Nisan et al. [41] as described in subsection 3.1. Nash equilibrium is a solution concept of non-cooperative game involving more than one player in which no player has an incentive to deviate from his or her chosen strategy after considering an opponent's choice. Overall, an individual can receive no incremental benefit from changing actions, assuming other players remain constant in their strategies.

Assume that *Assumption 1* (subsection 2.1.6) holds, so in particular the Wardrop's equilibrium (WE) v is unique. Let θ^k be the link toll vector of player $k \in K$, and let θ^{-k} denote a vector with all tolls in $K\backslash k$, i.e. $\theta^{-k} = (\theta^j,\, j \in K\backslash k)$. In the Nash game, for given $\bar{\theta}^{-k}$, the k^{th} stakeholder tries to find a solution toll $\bar{\theta}^k$ for the following problem:

$$\Psi^k(\bar{\theta}^k, \bar{\theta}^{-k}) \;=\; \min_{\theta^k} \Psi^k(\theta^k, \bar{\theta}^{-k})$$

where for given θ^k(and $\bar{\theta}^{-k}$)

$$\Psi^k(\theta^k, \bar{\theta}^{-k}) : \quad = \quad \min_{v^k} Z^k = C^k(v^k)$$
$$s.t$$

$$\begin{array}{rcl} \Lambda^T \left(\beta t(v^k) + \theta^k + \sum\limits_{j \,\epsilon\, K \backslash k} \bar{\theta}^j \right) & \geq & \Gamma^T \lambda^k \\ \left(\beta t(v^k) + \theta^k + \sum\limits_{j \,\epsilon\, K \backslash k} \bar{\theta}^j \right)^T v^k & = & \bar{d}^T \lambda^k \end{array} \quad and \quad \begin{array}{rcl} v^k & = & \Lambda f^k \\ \Gamma f^k & = & \bar{d} \\ f^k & \geq & 0 \\ (\theta^k & \geq & 0) \end{array} \tag{3.22}$$

For an elastic demand model where user actors and users take into account the consumer surplus, then system (3.22) is equivalent to:

$$\Psi_k(\theta^k, \bar{\theta}^{-k}) := \min_{v^k} Z_k = C^k(v^k) - \gamma_k \sum_{w \in W} \int_0^{d_w^k} B_w(\varsigma) d\varsigma \quad s.t$$

$$\begin{array}{rcl} \Lambda^T \left(\beta t(v^k) + \theta^k + \sum\limits_{j \in K \backslash k} \bar{\theta}^j \right) & \geq & \Gamma^T B(d^k) \\ \left(\beta t(v^k) + \theta^k + \sum\limits_{j \in K \backslash k} \bar{\theta}^j \right)^T v^k & = & B(d^k)^T d^k \end{array} \quad and \quad \begin{array}{rcl} v^k & = & \Lambda f^k \\ \Gamma f^k & = & d \\ f^k & \geq & 0 \\ (\theta^k & \geq & 0) \end{array}$$

The factor γ_k defines how much of the user benefit is considered in actor *k's* objective. The last condition on tolls $\left(\theta^k \geq 0\right)$ is necessary if the tolls are required to be non-negative.

A pure Nash equilibrium (NE) defines a situation where for all k it holds that: for fixed strategies $\bar{\theta}^{-k}$ of the opposing players, the best that player k can do is to stick to his own toll $\bar{\theta}^k$. A NE is thus a set of strategies/toll vectors $\bar{\theta} = (\bar{\theta}^k, k \in K)$ such that for each player k the following holds:

$$\Psi^k(\bar{\theta}^k, \bar{\theta}^{-k}) \leq \Psi^k(\theta^k, \bar{\theta}^{-k}) \qquad \text{for all feasible tolls } \theta^k \tag{3.23}$$

Observe that in the optimization problem above, each leader k can only change his own link toll vector θ^k. The strategies $\bar{\theta}^j, j \neq k$ of the other leaders are fixed in the $k's$ problem. The left hand constraints are the equilibrium conditions (see Eqs.(3.8) and (3.18)) and the right ones are the feasibility conditions.

3.3 Summary and conclusion

In this chapter, we have introduced and modelled a game theoretical approach to multi-objective optimization. Specifically, we have modelled road pricing game involving multiple actors in which each actor controls a traffic externality or objective that conflicts other actors' externalities/objectives. We derived the pricing schemes for all actors and defined the conditions for a Nash equilibrium. In the next chapter, we will describe the solution concepts of the approach.

Chapter 4

Solution concepts

Recall that we defined our road pricing game to be a game where various stakeholders and/or regions with different (and usually conflicting) objectives compete for toll setting in a given transportation network in order to satisfy their individual objectives or interests. In this chapter, we investigate classical game theoretical solution concepts for the road pricing game. We establish results for the road pricing game (discussed in subsection 3.1) so that stakeholders and/or regions confronted with such a game will know beforehand what is obtainable. This will save time and argument, and above all, get rid of the feelings of unfairness among the competing actors and road users. Among the classical solution concepts we investigate is Nash equilibrium. We show that the existence of a Nash equilibrium is not guaranteed in the game. In particular, we show that no pure Nash equilibrium exists among the actors, and further illustrate that even "mixed Nash equilibrium" may not be achieved in the road pricing game. The chapter also demonstrates the type of coalitions that are not only reachable, but also stable and profitable for the competing actors.

4.1 Existence of Nash equilibrium

4.1.1 Introduction

In game theory, it is often interesting to know if Nash equilibrium (NE) exists for non-cooperative games, and how to find it if it exists. Further, it will also be appealing to know if coalitions leave the players better off. In our road pricing game, NE translates to a tolling pattern that is stable among the stakeholders. Stability is used to mean a toll pattern where no stakeholder can improve his objective by changing his toll strategy given other players' toll pattern. If we can find a Nash toll pattern, then stakeholders can be presented with such a toll pattern since this will save them from time consuming debates and feelings of unfairness. Before we continue, we recall

Assumption 1 (in Chapter 2)**:**

- We assume throughout that the link cost (or travel time) function vector $t(v)$ is continuous and satisfies $(t(v) - t(\bar{v}))^T (v - \bar{v}) > 0 \;\forall v \neq \bar{v},\, v, \bar{v} \in V$ and all functions $C^k(v)$ are continuous, strictly convex, and strictly monotone (in the sense that $\partial C^k(v)/\partial v_a \geq 0 \;\forall k, a$), and the side constraints $g(v) \leq 0$ (see Eq.(2.1)), if used, are linear.

In this subsection, we investigate the existence of Nash equilibrium (NE) in our tolling game. We show below that this simple standard Nash equilibrium concept as described in the preceding Chapter (see Eqs.3.22 and 3.23) is not always applicable to the tolling problem. The main reason lies in the special structure of the problems $\Psi^k(\bar{\theta}^k, \bar{\theta}^{-k})$ in (3.22) leading to the following fact:

Fact: *Due to Assumption 1, for given vectors $\bar{\theta}^k, k \in K$ the corresponding solution $(\bar{v}, \bar{\lambda})$ of the system (3.22)* (i.e., *user equilibrium with respect to the costs* $\left[\beta t(v) + \sum_{j \in K} \bar{\theta}^j\right]$*) is unique.* Therefore:

Assertion: *If $\bar{\theta}$ is a Nash equilibrium toll vector, then all corresponding solution vectors $\left(\bar{v}^k, \bar{\lambda}^k\right)$ are identical for all actors, hence*

$$(\bar{v}^k, \bar{\lambda}^k) = (\bar{v}, \bar{\lambda}), \; k \in K. \tag{4.1}$$

Proof: Given that $\bar{\theta}^k$ solves problem (3.22) for all actors $k \in K$, then it means that at Nash equilibrium, the link toll vector $\bar{\theta}$ is given by $\bar{\theta} = \sum_{k \in K} \bar{\theta}^k$, where $\bar{\theta}_a = \sum_{k \in K} \bar{\theta}^k_a, \forall a \in A$. Due to *Assumption 1*, this toll vector $\bar{\theta}$ yields a unique flow pattern $\bar{v}$ and unique minimum route cost $\bar{\lambda}$. Of course, the users do not differentiate the tolls (per actor k), what they experience is the total toll vector $\bar{\theta}$, and as such, the vector $\bar{\theta}$ (together with the travel time costs) determines the unique user/Wardrop's equilibrium flow $\bar{v}$ and unique cost $\bar{\lambda}$ for the system. ■

4.1.2 Unrestricted toll values

From the relation (4.1) we can directly deduce the following result.

Corollary 3. *Suppose the leaders can toll all links with no restrictions (no constraint $\theta^k \geq 0$ in (Eq.(3.22))), then, for the tolling game, there does not exist a Nash equilibrium in general. Moreover, for this game, there is no stable coalition among players.*

Proof: Recall that (by Eq.(4.1)) the vector $(\bar{v}, \bar{\lambda})$ are the same for all actors at Nash equilibrium.

Assume that the actors' toll vector $\bar{\theta}$ is a Nash equilibrium toll with $(\bar{v}, \bar{\lambda}, \bar{\theta}^k)$ the solution of player k. Under the fact that at least one of the players, say player ℓ, has a different ideal (or optimal) link flow $\tilde{v}^\ell$ in SP_k (see Eq.(3.1)) since players are assumed to have conflicting objectives, and by our discussion in subsection 3.2.1 (see Eq.(3.19)), player ℓ can achieve this flow $(\tilde{v}^\ell)$ in $\Psi^\ell(\tilde{\theta}^\ell, \bar{\theta}^{-\ell})$ by choosing *e.g.*, the first-best pricing toll

$$\boxed{\tilde{\theta}^\ell = \nabla C^\ell(\tilde{v}^\ell) - \beta t(\tilde{v}^\ell) - \sum_{k \in K \setminus \ell} \bar{\theta}^k} \tag{4.2}$$

Since at any turn of the game (assuming now it is player ℓ's turn to play), player ℓ can toll $\tilde{\theta}^\ell$ as in (4.2) leading to his ideal flow $\tilde{v}^\ell$ in SP_ℓ (Eq.(3.1)), clearly no Nash equilibrium can be reached. Furthermore, since every actor ℓ can find a feasible $\tilde{\theta}^\ell$ as in (4.2), then there is no stable coalition among players since each actor $k \in K$ can always achieve SP_k on his own. ■

The same clearly holds for the case of elastic demand. Note that a link component of the toll vector $\tilde{\theta}^\ell$ given in (4.2) may be negative. In the next subsection, we show that the result of *Corollary 3* can be achieved even with restriction to non-negative tolls.

4.1.3 Restricted toll values

Corollary 4. *Even under the extra conditions $\theta^k \geq 0$ in Eq.(3.22), there does not exist a Nash equilibrium in general.*

Proof: For a fixed demand model, we can always achieve a first-best pricing toll in Eq.(3.20) satisfying $\tilde{\theta}^\ell \geq 0$: To see this, note that any leader $\ell \in K$ has the following valid toll vectors as part of a whole polyhedron (see proof below) that achieve the ideal flow vector $\tilde{v}^\ell$ in system (3.1) for leader ℓ:

$$\tilde{\theta}^\ell = \Big[\alpha\Big(\nabla C^\ell(\tilde{v}^\ell)\Big) - \beta t(\tilde{v}^\ell)\Big] - \sum_{k \in K\setminus\ell} \bar{\theta}^k; \quad \text{where } \alpha > 0 \tag{4.3}$$

By making α large enough (in view that C_ℓ is strictly monotonically increasing - see *Assumption 1*) we can assure $\tilde{\theta}^\ell \geq 0$. Again as in *Corollary 3*, at any point in the game, a player, say player ℓ, can toll $\tilde{\theta}^\ell$ as in (4.3) leading to his ideal flow $\tilde{v}^\ell$ in SP_ℓ (Eq.(3.1)), clearly no Nash equilibrium can be reached even with $\tilde{\theta}^\ell \geq 0$. ■

Proof of (4.3): Suppose $\tilde{v}^\ell$ is an ideal flow vector that solves (3.1) for player ℓ. Let θ^ℓ be the corresponding toll vector satisfying (3.21). By using the variational inequality transformation of the user equilibrium problem UE [77] - see (2.14) in subsection 2.1.6, it means that $\tilde{v}^\ell$ is a solution of the UP

$$\min_{v^\ell} \quad \Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)^T \quad v^\ell \qquad s.t. \qquad v \in V$$

where $\beta t(v)$ is a vector of link travel time functions. Obviously $\tilde{v}^\ell$ also solves the following UP:

$$\min_{v^\ell} \quad \alpha\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)^T \quad v^\ell \qquad s.t. \qquad v \in V \qquad \text{where } \alpha > 0$$

but,

$$\begin{aligned}
\alpha\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)^T v^\ell &= \Big(\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big) + (\alpha - 1)\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)\Big)^T v^\ell \\
&= \Big(\beta t(\tilde{v}^\ell) + \Big[\theta^\ell + (\alpha - 1)\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)\Big]\Big)^T v^\ell
\end{aligned}$$

this means that with θ^ℓ, any vector

$$\tilde{\theta}^\ell \;=\; \Big[\theta^\ell + (\alpha - 1)\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big)\Big] = \alpha\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big) - \beta t(\tilde{v})$$

is a valid toll vector as well. Recall that for one objective C^ℓ, the marginal social cost (MSC) toll given by Eq.(3.19)

$$\theta^\ell \;=\; \nabla C^\ell(\tilde{v}^\ell) - \beta t(\tilde{v}^\ell)$$

is one toll vector that achieves the ideal flow vector $\tilde{v}^\ell$, therefore

$$\begin{aligned}
\tilde{\theta}^\ell &= \alpha\Big(\beta t(\tilde{v}^\ell) + \theta^\ell\Big) - \beta t(\tilde{v}^\ell) = \alpha\Big(\beta t(\tilde{v}^\ell) + \Big(\nabla C^\ell(\tilde{v}^\ell) - \beta t(\tilde{v}^\ell)\Big)\Big) - \beta t(\tilde{v}^\ell) \\
&= \alpha\Big(\nabla C^\ell(\tilde{v}^\ell)\Big) - \beta t(\tilde{v}^\ell)
\end{aligned}$$

In the presence of other actors' toll $\sum_{k\epsilon K\backslash\ell}\bar{\theta}^k$, $\tilde{\theta}^\ell$ now becomes

$$\tilde{\theta}^\ell \;=\; \alpha\Big(\nabla C^\ell(\tilde{v})\Big) - \beta t(\tilde{v}) - \sum_{k\in K\backslash\ell}\bar{\theta}^k; \quad \text{where } \alpha > 0 \qquad \blacksquare$$

Equation (4.3) suggests that the tolls could grow infinitely large as a result of actors' move to achieve their ideal or optimal objective values (see the example on the non-existence of NE in chapter 4 subsection 4.3.1). Such a high toll, though theoretically possible due to fixed demand, is, in fact, not realistic since high tolls may discourage some users from travelling or at least make them change their mode of transportation. This phenomenon is captured when demand is allowed to be elastic; when tolls are restricted to be non-negative, and demand assumed elastic, a very high toll pattern implies that OD demands will near zero, which in turn lowers the societal welfare or economic benefit of the actors as described in the actors' objectives (see for example, the objective of system (2.1)). Further, from Eq.(2.17), we have that for any given flow pattern $(\hat{v}, \hat{d})$, the total network toll is given by

$$\theta^T\hat{v} = B(\hat{d})^T\hat{d} - \beta t(\hat{v})$$

revealing that the link toll vector θ is bounded. Note that the boundedness of the tolls in elastic demand does not guarantee the existence of Nash equilibrium (see Braess example below).

We emphasize that extra restrictions on the tolls θ^k may play in favour of the existence of a Nash equilibrium as we will demonstrate with examples.

In general, what can we say about the existence of NE? A well-known theorem in game theory [65] states that a game has a Nash equilibrium if the following conditions are met:

- the strategy sets for each player are compact and convex, and
- each player's cost function $\Psi^k(\theta^k, \bar{\theta}^{-k})$ is continuous and quasi-convex in his strategy θ^k.

However, in general, we cannot expect such a convexity property. Even the mostly used "system optimization" function, the travel time function, is in general not convex as we will show with an illustrative example (see Braess network example below).

Since we do not expect a Nash equilibrium to exist in general for the road pricing game, it means that in practice, rational stakeholders or actors may never reach an agreement on a given toll pattern. This may be an indication why road pricing, even with its rich potentials in alleviating a lot of traffic externalities, is still unpopular among stakeholders and road users. In most countries, like the United States (New York City in 2008) and The Netherlands (in 2011), the road pricing scheme was almost at implementation stages when the parliament withdrew the idea due to conflicts of interests.

4.2 Numerical examples I

4.2.1 The Braess network example

We use a well-known network to show that even the total network travel time (as an objective), in general, may not be convex in the tolls (strategy set). Such a drawback may lead to non-existence of Nash equilibrium in the road pricing game [65] (see our example below). The yellow labels in *figure 4.1* are the unique link identities, numbering the links from 1 to 5. The other labels are the costs a user encounters by using the links (for example, v_2 for *link 2* and $v_4 - 0.5v_4^2$ for *link 4*, where v_i is the flow on link i). The fixed demand from node a to node d is 1. $\theta_i \in [0, 1]$ represents the toll on link i where $\theta_i = 0$, for $i \neq 1, 3$. We have grouped all possible tolls into two classes, namely ; $\theta_3 \leq \theta_1$ and $\theta_3 \geq \theta_1$, and derive the following user equilibrated flows v_i on the links:

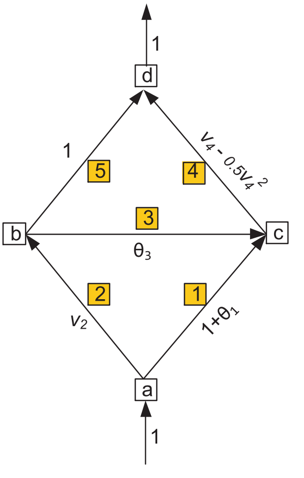


Figure 4.1: The Braess' network

for $\theta_3 \leq \theta_1$

if $\theta_3 \leq 0.5$, then $v_1 = v_5 = 0,\ v_2 = v_3 = v_4 = 1$
if $\theta_3 \geq 0.5$, then $v_1 = 0,\ v_2 = 1,\ v_3 = v_4 = 1 - (1 + 2(\theta_3 - 1))^{1/2}$,
$v_5 = (1 + 2(\theta_3 - 1))^{1/2}$

for $\theta_3 \geq \theta_1$

if $\theta_3 \leq 0.5$, then $v_1 = (\theta_3 - \theta_1),\ v_2 = v_3 = 1 - (\theta_3 - \theta_1),\ v_4 = 1,\ v_5 = 0$
if $\theta_3 \geq 0.5$

$$v_1 = \begin{cases} (\theta_3 - \theta_1) & if\ (\theta_3 - \theta_1) \leq 1 - (1 + 2(\theta_3 - 1))^{1/2} \\ 2 - 2(0.5 + 0.5\theta_1)^{1/2} & otherwise \end{cases}$$

$$v_2 = \begin{cases} 1 - (\theta_3 - \theta_1) & if\ (\theta_3 - \theta_1) \leq 1 - (1 + 2(\theta_3 - 1))^{1/2} \\ 2(0.5 + 0.5\theta_1)^{1/2} - 1 & otherwise \end{cases}$$

$$v_3 = \begin{cases} 1-(1+2(\theta_3-1))^{1/2}-(\theta_3-\theta_1) & if\ (\theta_3-\theta_1) \leq 1-(1+2(\theta_3-1))^{1/2} \\ 0 & otherwise \end{cases}$$

$$v_4 = \begin{cases} 1-(1+2(\theta_3-1))^{1/2} & if\ (\theta_3-\theta_1) \leq 1-(1+2(\theta_3-1))^{1/2} \\ 2-2(0.5+0.5\theta_1)^{1/2} & otherwise \end{cases}$$

$$v_5 = \begin{cases} (1+2(\theta_3-1))^{1/2} & if\ (\theta_3-\theta_1) \leq 1-(1+2(\theta_3-1))^{1/2} \\ 2(0.5+0.5\theta_1)^{1/2}-1 & otherwise \end{cases}$$

Let the tolls now satisfy $\theta_3 \geq \theta_1\, and\, \theta_3 \geq 0.5\, and\, (\theta_3-\theta_1) \leq 1-(1+2(\theta_3-1))^{1/2}$, then the system travel time function $v^T t(v)$ is given by:

$v^T t(v) = v(\theta)^T t(v(\theta)) = 1.5-(\theta_3-\theta_1)+(\theta_3-\theta_1)^2+0.5(1+2(\theta_3-1))^{1/2}-0.5(1+2(\theta_3-1))+0.5(1+2(\theta_3-1))^{3/2}$.

Note that we follow the traditional way of modelling travel time function in which the tolls are not optimized in $v^T t(v)$, so, for example, the travel time for the object $v^T t(v)$ on *link 1* is $v_1(1+0) = v_1$, and that of *link 3* is $v_3 \cdot 0 = 0$. We assume that the tolls are returned back into the transportation system so as not to increase transportation costs.
The Hessian of the *travel time (TT)* function $v^T t(\theta)$ is given by

$$H_{TT} = \begin{bmatrix} 2 & -2 \\ & \\ -2 & 2+\frac{3}{2}(1+2(\theta_3-1))^{-(1/2)}-\frac{1}{2}(1+2(\theta_3-1))^{-(3/2)} \end{bmatrix}$$

The major determinant of this matrix is negative if $\theta_3 \in \left(\frac{1}{2},\, \frac{2}{3}\right)$, thus, we conclude that the travel time function $v^T t(v)$ is in general not convex in the strategy set $\{\theta_1, \theta_3\}$. So, we do not guarantee the existence of a Nash equilibrium for the road pricing game. This non-convexity property does not change even when other players' objectives are convex in their strategy sets [65]. In fact, this example is simply to illustrate that the existence of Nash equilibrium of the road pricing game may also depend on *how the objectives of optimization are defined.* Furthermore, the example reveals that though bounding the tolls (as we claimed earlier) may help the game converge to a Nash equilibrium, it is, in fact, not sufficient for the Nash convergence.

4.2.2 Two-node network example

In this subsection, we bound the tolls and use a simple example to illustrate how changes in cost functions (on the same network) affect the existence of Nash equilibrium (NE). We further demonstrate that the *existence of NE may be sensitive to the constant operational cost of the toll booths.*

In this example, we consider a network of two links; a and b, and two actors; *actor I* and *actor II*. Actors are respectively interested in minimizing two different types of "*traffic*" costs C^I and C^{II} for the network. We use χ^I and χ^{II} to describe the link cost (negative utility) functions for the "*traffic*" costs C^I and C^{II} respectively. We also denote by OC the *operational cost* for a toll booth.

$$\chi^I = \begin{cases} \chi_a^I = \begin{cases} 2v_a & if\, \theta_a^I = 0 \\ 2v_a + OC & otherwise \end{cases} & for\, link\, a \\ \chi_b^I = \begin{cases} 2.5v_b & if\, \theta_b^I = 0 \\ 2.5v_b + OC & otherwise \end{cases} & for\, link\, b \end{cases}$$

$$\chi^{II} = \begin{cases} \chi_a^{II} = \begin{cases} 2v_a + 2 & if\, \theta_a^{II} = 0 \\ 2v_a + 2 + OC & otherwise \end{cases} & for\, link\, a \\ \chi_b^{II} = \begin{cases} 3v_b & if\, \theta_b^{II} = 0 \\ 3v_b + OC & otherwise \end{cases} & for\, link\, b \end{cases}$$

$$C^I = e^T \chi^I \text{ and } C^{II} = v^T \chi^{II}$$

where $v_a + v_b = 2$, θ_i^k is the toll of player $k \in \{I, II\}$ on link $i \in \{a, b\}$, $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$

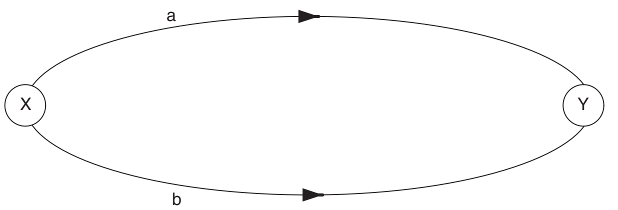


Figure 4.2: Two-node network

We set $OC = 0.55$ *per toll booth*. We take χ^{II} $\left(\text{when } \theta_a^{II} = \theta_b^{II} = 0\right)$ to represent the travel time function of the links, in other words, *actor II* cares for *travel time* cost of the system $\left(C^{II} = v^T \chi^{II} + OC\right)$, and of course, the *operational cost* if he wants to set a toll (booth) on a link. Actors can only choose strategies from the following set of discrete toll: $\theta_i^k \in \{0, 1, 2, 3, 4, 5, 6\}\ \forall k, i$. The tolling game involves the two actors choosing tolls (in turns) from the set of the feasible tolls and setting them on links to optimize their individual objectives. Note that in each turn, an actor may decide to reduce, add or remove a toll on any of the links. For each move by an actor, the resulting flow must be in User Equilibrium (UE).

Outcome of the tolling game

$v = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$, $SC = (Total)\, System\, Cost = C^I + C^{II}$

Table 4.1: Outcome of the Two-player Road Pricing Game

User equilibrium or no toll scenario

Link	Tolls	v	OC	C^I	C^{II}	Path cost	SC
a	(0,0)	0.8	(0,0)	1.6	2.9	3.6	4.5
b	(0,0)	1.2	(0,0)	3	4.3	3.6	7.3
				4.6	**7.2**		**11.8**

Player I goes first leading to

Link	Tolls	v	OC	C^I	C^{II}	Path cost	SC
a	(0,0)	2	(0,0)	4	12	6	16
b	(6,0)	0	(0.55,0)	0.55	0	6	0.55
				4.55	**12**		**16.55**

then player II, leading to

Link	Tolls	v	OC	C^I	C^{II}	Path cost	SC
a	(0,5)	1	(0,0.55)	2	4.55	9	6.55
b	(6,0)	1	(0.55,0)	3.05	3	9	6.05
				5.05	**7.55**		**12.6**

then player I again, leading to

Link	Tolls	v	OC	C^I	C^{II}	Path cost	SC
a	(0,5)	0	(0,0.55)	0	0.55	7	0.55
b	(0,0)	2	(0,0)	5	12	6	17
				5	**12.55**		**17.55**

then player II's best strategy now is to withdraw all its toll, leading to the first table (the user equilibrium table), and this creates a cycle.

The first table in *table 4.1* represents the user equilibrium (UE) on a toll free network. The UE flow w.r.t χ^{II} on links a and b are respectively *0.8* and *1.2.* Note that in *table 4.1* , the tolls as well as OC on the links are represented in a vector form $(x,\ y)$, where the first entry belongs to *player I,* and the second entry to *player II*. *SC* is the social or system cost which represents the total cost experienced in the system excluding the tolls (since the tolls are assumed to be returned into the transportation network in one form or the other). An actor can add or remove tolls depending on which strategy optimizes his objective. In the second table, that is, *the first move by actor I*, he (*actor I*) sets his maximum possible link toll (i.e. 6) on link b to shift traffic to link a (see C^I). Observe the cost of *0.55* on link b under OC for *player I* indicating the cost of operating the toll booth. Under the columns C^i, are the total link costs for *players I* and *II,* the boldfaced numbers are the total network cost for the players. The total system cost (SC) is also in bold; for instance, in the *first move of player II* (third table) where $v_a = v_b = 1$, $C_a^{II} = v_a\chi_a^{II} + OC = v_a(2v_a + 2) + OC = 1(2 * 1 + 2) + 0.55 = 4.55$, and $C_b^{II} = v_b\chi_b^{II} = v_b\,(3v_b) = 1\,(3 * 1) = 3$, and $C^{II} = C_a^{II} + C_b^{II} = 4.55 + 3 = \mathbf{7.55}$. Notice that operational cost $OC = 0$ unless a player has set a toll (booth) on a link (in that case, $OC = 0.55$). The path cost corresponding to the third table is calculated as follows $path\,cost\,(link\,a) = \chi_a^{II} + toll\,on\,link\,a = (2v_a + 2) + toll = 4 + 5 = 9$, similarly $path\,cost\,(link\,b) = \chi_b^{II} + toll\,on\,link\,b = 3v_b + toll = 3 + 6 = 9.$

A precise analysis reveals that this game has no Nash equilibrium (NE), and the actors will perpetually have the incentive to change their strategies.

Remarks about this game

1. The system optimal outcome for the problem above is given by:

$$v_a = 0.95, v_b = 1.05, C^I = 4.47, C^{II} = 7.01, SC = 11.48.$$

2. In general, the Nash equilibrium solution boundary of the game is given as follows:

   for OC : $\quad 0.5 < OC < 0.6 \quad NE\,does\,not\,exist, \quad otherwise\,NE\,exists$

   This shows that the existence of NE can depend on the cost of operating the toll booths.

3. If we ignore the OC in the model, then NE always exist among the two actors. In this case, all actors use up to their maximum toll with $\theta_a = (0, 6)$, and $\theta_b = (6, 0)$. Without boundary restriction on the link tolls, the two leaders will infinitely keep on increasing the link tolls ($actor\ I$ on links b and II on link a). This is true since the demand is fixed. In fixed demand models, it is assumed that the fixed number of trips must be realized irrespective of the cost of travel. On the other hand, with elastic demand, infinite tolls by the leaders will, of course, result in no travel scenario which in turn results to zero benefit for the actors (see section 2.1.6). As such, infinite positive toll vectors are not possible when demand is elastic.

4. By mere interchanging the link cost functions of actor I, that is,

$$\chi^I = \begin{cases} \chi_a^I = \begin{cases} 2.5v_a & if\,\theta_a^I = 0 \\ 2.5v_a + OC & otherwise \end{cases} & for\,link\,a \\ \chi_b^I = \begin{cases} 2v_b & if\,\theta_b^I = 0 \\ 2v_b + OC & otherwise \end{cases} & for\,link\,b \end{cases}$$

   Nash equilibrium exists for any value of OC.

5. The cost function of the type described in this example (which includes the cost of operating the toll booths) has a good practical bases. This means that in practice, Nash equilibrium may not exist for the road pricing game.

4.2.3 Mixed Nash equilibrium for the game in Table (4.1)

For the two-player cost minimization game described in subsection 4.2.2 above, turn by turn, the players choose the following toll strategies during the game: (0,0) on the links (a,b) or (0,6) on the links (a,b) for *player I*, and (0,0) on the links (a,b) or (5,0) on the links (a,b) for *player II* (see the "Tolls" column of *table 4.1* or the matrix under the heading "interpretation" below). Every toll strategy translates to a feasible flow patter, which in turn translates to cost for the players, the matrix representation of the two-player cost game is thus given by:

$$\begin{array}{cc} & \begin{array}{cc} q & 1-q \end{array} \\ \begin{array}{c} p \\ \\ 1-p \end{array} & \begin{pmatrix} 4.55, 12.00 & 5.05, 7.55 \\ & \\ 4.60, 7.20 & 5.00, 12.55 \end{pmatrix} \end{array}$$

Observe of course that the game has no pure NE as we have seen in subsection 4.2.2. In the mixed strategy game, *player I* has the strategy (p, *1-p*) of playing (*Top, Bottom*) and *player II*, the strategy (q, *1-q*) of playing (*Left, Right*), where p and q are probabilities. The best reply functions for both players are given below:

$$\beta_I(q) = \begin{cases} \{(1,0)\} & if \ \frac{1}{2} < q \leq 1 \\ \{(p, 1-p) | 0 \leq p \leq 1\} & if \ q = \frac{1}{2} \\ \{(0,1)\} & if \ 0 \leq q < \frac{1}{2} \end{cases}$$

$$\beta_{II}(p) = \begin{cases} \{(1,0)\} & if \ 0 \leq p < \frac{5.35}{9.8} \\ \{(q, 1-q) | 0 \leq p \leq 1\} & if \ p = \frac{5.35}{9.8} \\ \{(0,1)\} & if \ \frac{5.35}{9.8} < p \leq 1 \end{cases}$$

Graphically:

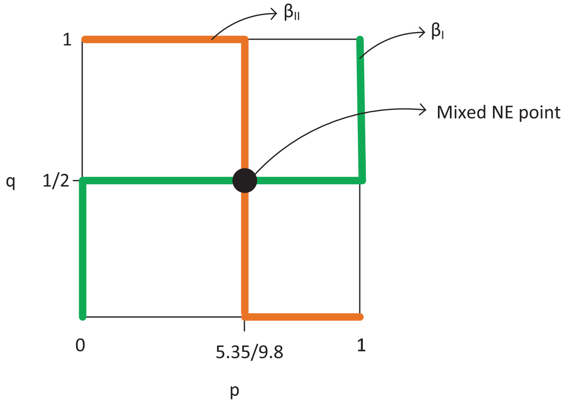


Figure 4.3: Graphical representation of the mixed Nash equilibrium

The Mixed Nash equilibrium point (p, q) corresponds to ($\frac{5.35}{9.8}, \frac{1}{2}$). So, the strategies of *player I* and *player II* that will lead to mixed Nash equilibrium are ($\frac{5.35}{9.8}, \frac{4.45}{9.8}$)

and $(\frac{1}{2}, \frac{1}{2})$, respectively. The expected cost is $C^I = 4.80$, $C^{II} = 9.82$, $SC = 14.62$.

Interpretation: A (mixed) Nash equilibrium exists between the players if we can find (mixed) strategy tolls among the actors such that the equilibrium cost point $C^I = 4.80$, $C^{II} = 9.82$ is reached. In other words, with probability p actors now choose random link tolls $\theta = p \cdot 0 + (p-1) \cdot 6$ instead of $\theta \in \{0, 1, 2, 3, 4, 5, 6\}$. Thus, for any probability choice p of an actor in the mixed strategy, a toll $\theta = p \cdot 0 + (p-1) \cdot 6$ results, and further, we also have a corresponding expected cost $C^i(\theta)$ for this actor. Note that $C^i(\theta)$ is not linear in the strategy θ.

The following tolls strategy matrix with the associated probabilities $(\hat{p}, \hat{q})$ for *players I* and *II* (the right matrix is the cost matrix) should translate to the mixed Nash equilibrium cost $C^I = 4.80$, $C^{II} = 9.82$, $SC = 14.62$ deduced above:

$$\begin{array}{cc} & \begin{array}{cc} \hat{q} & 1-\hat{q} \end{array} \\ \begin{array}{c} \hat{p} \\ \\ 1-\hat{p} \end{array} & \begin{pmatrix} (0,6),(0,0) & (0,6),(5,0) \\ \\ (0,0),(0,0) & (0,0),(5,0) \end{pmatrix} \end{array} ; \quad \begin{array}{cc} & \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array} \\ \begin{array}{c} \frac{5.35}{9.8} \\ \\ \frac{4.45}{9.8} \end{array} & \begin{pmatrix} 4.55, 12.00 & 5.05, 7.55 \\ \\ 4.60, 7.20 & 5.00, 12.55 \end{pmatrix} \end{array}$$

Recall that Nash [40] proved in his famous theorem that every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one mixed (Nash) equilibrium. In contrary to what Nash said, our game has just two players and two possible actions per player, yet we cannot find a pair of strategy $(\hat{p}, \hat{q})$ such that the resulting cost is in equilibrium. The explanation lies on the fact that the expected costs $C^I = 4.80$, $C^{II} = 9.82$ is not linear in the strategy $(\hat{p}, \hat{q})$ (recall the Braess's example in subsection 4.2.1), and thus, the cost functions for the players under mixed strategy $(\hat{p}, \hat{q})$ may not be continuous or quasi-convex [65].

Each pair of entries represents toll actions of *players I* and *II* respectively. The first entry of each action is a player's toll action on link *a,* and the second entry, his action on link *b*. For example, the Top-Right entry of the toll matrix has the entry $(0, 6), (5, 0)$ and this means that *player I* has no toll on *link a* and tolled *6* on *link b*, while *player II* tolled *5* on *link a* and nothing on *link b* (see the "Tolls" column of third table of table 4.1). In the mixed strategy model, with probability $(\hat{p}, \hat{q})$ the cost is defined by $\theta = (0, 6), (0, 0); (0, 6), (5, 0); (0, 0), (0, 0); (0, 0), (5, 0)$ (see the matrix above). In other words, this is a linearisation of the real costs. It turns out that there is no choice probability $(\hat{p}, \hat{q})$ for the actors in the toll matrix game such that the resulting mixed tolls θ leads to the mixed NE of the cost matrix game $C^I = 4.80$, $C^{II} = 9.82$. This is because, there is no (feasible) flow vector (v_a, v_b) such that the mixed equilibrium costs $C^I = 4.80\,(\pm 0.55)$, and $C^{II} = 9.82\,(\pm 0.55)$ hold at the same time. This shows that even with a finite number of players, and finite number of turns, we cannot find a mixed toll vector such that Nash equilibrium exists among the actors. In fact, the strategy $(\hat{p}, \hat{q})$ has no direct implication on the cost (utility) matrix since

1. The cost functions are generally not convex in the toll strategies of the players (see [43])
2. The tolls in the toll matrix are not uniquely determined, and this means that the strategy $(\hat{p}, \hat{q})$ is not uniquely given, and

3. The non-unique tolls lead to a unique Wardrop's equilibrium which determines the cost matrix, and
4. We also acknowledge that the traffic flow is not linear.

4.3 Numerical examples II

In this section, we use numerical examples to demonstrate the models developed in this thesis. First, we start by demonstrating how bad the social transportation welfare could be if only one or part of the traffic externalities that affect the transportation network is optimized. It demonstrates the need for multi-objective optimization of the traffic externalities when designing road pricing schemes. In the second part of the example, we demonstrate the game theoretic approach to the road pricing game where each actor controls a specific objective. Note that our game model generalises a set-up where different subsets of an entire network is controlled by various actors usually with conflicting interests. We first show that cooperation among the (competing) actors will lead to a socially desirable toll pattern (and hence the flow pattern), but as we stated earlier, with autonomous actors, why would they agree to cooperate if they can achieve a better outcome on their own or by at least forming partial coalitions among themselves? To this, we further demonstrate the results of the non-cooperative (or Nash equilibrium) game among the actors. In general, competition deteriorates the social welfare of a system, but the fact is that some stakeholders may be far better off competing with others than coalescing with them. Recall that with our optimal Nash inducing mechanism, we can induce a desired toll pattern among the competing actors in a way that the result of the non-cooperative game is the socially desired outcome. We then show how users' interests could be represented in the upper level of the game, in particular, users are represented by one actor who "lobbies" for an alternative but a lower toll pattern that still guarantee other actors their Nash or cooperative outcome. The aim of this is to demonstrate that users' interests (for example, *making sure that the link tolls stay as low as possible*) can be represented during toll decision making process. A scheme with such an extra "condition" on tolls may seem more appealing to the road users than the one without such a condition (see the example below). Finally, we demonstrate our results in Ohazulike et al. [44] on the existence of the Nash equilibrium toll for bounded tolls.

4.3.1 Five-node network example

We will use a five-node network to illustrate the models developed so far in this thesis.

Link attributes and input

We have used the following cost functions:

System Travel Time Cost: $C^t(v) = \sum_{a\in A} \beta v_a t_a(v_a) = \sum_{a\in A} \beta v_a T_a^{ff}\left(1+\eta\left(\frac{v_a}{C_a}\right)^{\phi}\right)$;

the so called *Bureau for Public Roads (BPR) function,* where
T_a^{ff} - free flow travel time on link a,

v_a - total flow on link a,
$\hat{C}_a$ - practical capacity of link a,and
$\eta\, and\, \phi$ - BPR scaling parameters.
We will use $\eta = 0.15$, $\phi = 4$ and β (value of time - VOT) = €$0.1671667/minute$, see Table 4.2 for other parameters.

Emission Cost: $C^e(v) = \sum\limits_{a \in A} v_a \alpha_a \varkappa_a l_a$; where
$\varkappa_a$ - emission factor for link a (depending on the emission type and the vehicle speed on link a in g/vehicle-kilometre).
l_a - length of link a. In this case study, we only consider two emission types; NO_x and PM_{10}.
See Table 1a for the emission costs α_a and 1c for the emission factor $\varkappa_a$.

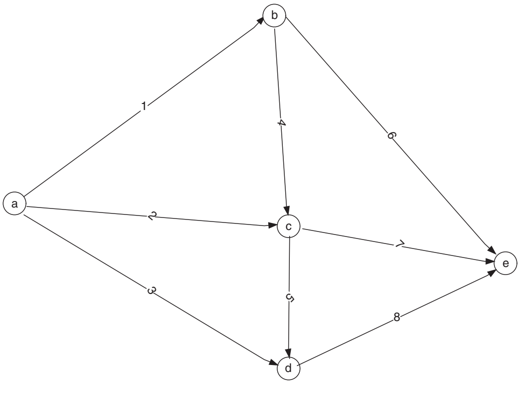


Figure 4.4: The Five-node network

Noise Cost: $C^n(v) = \sum\limits_{a \in A} \gamma \left[A + B \log(\frac{v_a}{v_0}) + 10 \log(\frac{v_a}{v_a}) \right] h_a$; where
A and B in $dB(A)$ - vehicle specific constants as given in [29].
$v_a\, and\, v_0(v_{ref})$ - the average and reference speed of vehicles on link a respectively.
h_a - number of households along link a.
We will use the widely used parameter values *A=69.4dB(A), B=27.6dB(A)* and $v_0(v_{ref}) = 80km/hr$ [51], see Table 4.2 for h_a and the monetary conversion parameter γ.

Infrastructure (Pavement) Cost: $C^i(v) = \sum\limits_{a \in A} v_a \tau_a (\frac{H_a}{J_a}) l_a$; where
τ_a - load equivalence factor (LEF) that measures the amount of pavement deterioration produced by each vehicle on link a, measured in €/vehicle-kilometre. We will set $\tau_a =$ **€**$0.0024/veh - km$ for all links.
H_a - initial cost for the infrastructure per kilometre.
J_a - design standard of link a measured by the design number of equivalent axle load (ESAL) repetitions.
$\frac{H_a}{J_a}$ - unit investment cost per ESAL-kilometre. The higher the design standard of an infrastructure, the smaller the factor $(\frac{H_a}{J_a})$, meaning that infrastructure with a high design standard are the most cost-effective way to handle high traffic volumes [52].
We will use $\frac{H_a}{J_a} = 1$.

Safety Cost: $C^s(v) = \sum_{a \in A} v_a \varrho \kappa_a l_a$; where
κ_a - risk/safety factor for link a, measured in number of injury-crashes/vehicle-kilometre.
$l_a * v_a$ - measure of level of exposure on link a.
We will set cost of one injury $\varrho = €300/injury$.

Toll Revenue: $C^{TR}(v) = \sum_{a \in A} v_a \theta_a$; where
θ_a - is the toll on link a.

We will omit the last objective in our analysis.

Emission factors are from the CAR-model [25], safety factors, emission costs and injury costs are chosen in a reasonable way (see below), and noise costs are from [70]. The value of time (VOT) used is as stated in [4]. MATLAB is used to solve the programs.

We have for the OD pair $w = (a - e)$, that the fixed travel demand $\bar{d}$ is given by:

$$\bar{d} = 1000 \tag{4.4}$$

where $\bar{d}_w$ is the fixed OD demand for the w^{th} OD.

Table 4.2: Link attributes and characteristics

Link Attributes (vehicle class: private cars)

Links	Length (km)	Free Speed (km/hr)	Link capacity	# of households around the links	Emission cost NO_x(€/gram)	Emission cost PM_{10}(€/gram)	Safety factor (injury /veh-km)
1	10	100	400	1400	10	5	0.008
2	7	70	300	2000	10	5	0.08
3	10.5	100	350	3000	45	40	0.008
4	5	70	200	200	60	60	0.00001
5	4	70	250	200	45	40	0.00001
6	10	90	250	2500	10	5	0.09
7	5	80	250	2800	10	5	0.009
8	8.5	90	300	1800	45	40	0.009

Cost of noise per household as measured from road traffic (Euro per year 2007 price scale)

dB(A)	< 55	55 - 65	66 - 75	> 75
Euro per dB(A)	0	27	40	45.4

Emission factors (g/km/veh)

Speed (km/hr)	NO_x	PM_{10}
< 15	0.702	0.061
≤ 30	0.456	0.059
≤ 45	0.48	0.059
< 65	0.227	0.035
≥ 65	0.236	0.043

4.3.2 Result

In this subsection, we have shown the results of the models developed so far in this thesis when applied to a simple network. In particular, we demonstrate the effect of single objective optimization in a system that comprises of multiple conflicting objectives. Further, in order to solve the multiple objective problem (MOP), we utilize the game models developed so far in this thesis to solve the MOP, demonstrating that the objectives can be improved for each actor participating in the road pricing game. Interestingly, users represented by a stakeholder in the upper level could achieve a lot for the road users by participating in the road pricing game.

Table 4.3 shows the ideal link and path flows, and costs of objectives when various objectives are singly optimized (as in Eq.3.1) and in an aggregated multi-objective - MO form (as in Eq.2.23 or 4.5). The table shows how other externalities as well as the societal welfare are adversely affected when only one or part of the externalities is optimized. It describes what happens when one stakeholder with selfish interest controls the affairs of the transportation network. In the presence of more than one actor who determines the tolling scheme, it means then that none of these ideal link flows (and the corresponding ideal tolls) as given in Table 4.3 is likely to be achieved since the objectives are conflicting. In table 4.3, UE displays the Wardropian equilibrium on a toll free network. Table 4.3b displays the (non-unique) path flows corresponding to table 4.3a. Table 4.3c displays the effect of single objective optimization on the system cost (see the last column) and other objectives. Table 4.3c shows how single objective optimization can adversely affect other objectives (observe the very high cost entries) and the system. The system cost is of course minimal when the objectives are optimized in an aggregated form. This can be seen from the last entry of the last column of table 4.3c (see also Eq.2.23). The bold diagonal entries are the optimal objective values (corresponding to the ideal link flows of table 4.3a) for the single objective optimization problems. Note that the system/social cost (SC or MO) is defined as in Eqs.(2.23) or explicitly in (4.5) below.

$$\begin{aligned} SC = MO \quad = \quad & \sum_{a\in A}\Big(\beta v_a t_a(v_a) + \alpha_a v_a \varkappa_a l_a + \gamma\left[A + B\log(\frac{\upsilon_a}{\upsilon_0}) + 10\log(\frac{v_a}{\upsilon_a})\right] h_a \\ & + \tau_a v_a(\frac{H_a}{J_a}) l_a + \varrho v_a \kappa_a l_a\Big) \end{aligned} \tag{4.5}$$

In fact, the objective values in table 4.3c are derived from the input functions for those objectives.

Cooperative and non-cooperative leaders' game

In this subsection, we demonstrate the road pricing game among several actors using the five-node network. For clarity, we only consider three actors whose interests are respectively to minimize their own costs. These actors have the following objectives: *system travel time cost* $(C^t(v))$, *emission cost* $(C^e(v))$ and *safety cost* $(C^s(v))$, respectively. We denote these actors by "t", "e" and "s", respectively. We assume non-negative link tolls and toll bound of *[0,5]EUR* per link per player (to stimulate convergence or the existence of Nash equilibrium).

Single leader (multi-objective) road pricing

Cooperative game

For comparison reasons, we have shown the result of a cooperative game among the three actors (see table table 4.4a). In the cooperative game, the actors coalesce and solve the fixed demand version of the multi-objective models of system (2.23) using (4.5) (aggregating the three costs). Here the actors search for a common flow vector that will minimize their collective costs. With this optimal flow pattern, they now search for a common toll pattern that will yield the optimal flow pattern as a user equilibrium flow. It turned out that the first-best toll vectors could be found with the toll bound of *[0,15]EUR* per link (recall the bound of *[0,5]EUR* per link per player).

Table 4.3: Link flows, path flows and cost for the single leader game

4.3a: Link flows when the objectives are optimised singly and as multi-objective(MO)

Objectives –> Links	UE	Travel Time	Emission	Noise	Safety	Infrastructure	MO
1	281	320	601	995	1,000	0	261
2	369	324	399	4	0	1,000	319
3	350	356	0	1	0	0	420
4	0	16	0	993	1,000	0	261
5	0	0	22	993	0	0	0
6	281	304	601	2	0	0	0
7	369	340	377	3	1,000	1,000	580
8	350	356	22	995	0	0	420

4.3b: Corresponding path flows

Path flows (f_r)–> Paths (r)	UE	Travel Time	Emission	Noise	Safety	Infrastructure	MO
a–b–e	281	304	601	2	0	0	0
a–b–c–e	0	16	0	0	1,000	0	261
a–b–c–d–e	0	0	0	993	0	0	0
a–c–e	369	324	377	3	0	1,000	319
a–c–d–e	0	0	22	1	0	0	0
a–d–e	350	356	0	1	0	0	420
Total OD demand	1,000	1,000	1,000	1,000	1,000	1,000	1,000

4.3c: Single objective (and MO) optimization and the corresponding cost effect on other objectives and on the system (€)

Cost effects–> Objectives	UE	Travel Time	Emission	Noise	Safety	Infrastructure	System Cost
UE	**2,001**	2,421	107,336	10,678	166,460	40	286,936
Travel Time	2,015	**2,388**	110,570	10,568	165,979	41	289,547
Emission	2,778	6,122	**58,698**	7,775	249,288	41	321,923
Noise	24,988	112,017	723,355	**3,035**	48,063	66	886,537
Safety	21,769	99,460	338,775	3,688	**37,515**	48	479,487
Infrastructure	10,159	44,276	87,900	3,977	181,500	**29**	317,682
MO	2,391	4,065	142,726	8,744	87,875	41	**243,451**

One-shot non-cooperative game

Here, we describe what happens when autonomous actors or stakeholders or local authorities optimize their individual objectives without any knowledge of what other actors are doing. In this case, actors may be aware of the other actors, but they do not know what their objectives look like. They act solely to optimize their individual objectives, but then, the outcome on the network will be as a result of their cumulative actions. For example, in the toll setting problem, without any knowledge of other actors' toll vectors, actors set tolls in one shot to optimize their individual objectives (see Eq.3.1). Each actor assumes he is the only leader and sets his ideal toll that would lead to his optimal flow as in single leader game. Again, with the toll bound of [0,5]EUR per link per player, all actors proposed his first-best toll except for actor "s" who proposed his "second-best" toll since there is no feasible toll pattern within this toll bound that enables his optimal flow pattern. In classical game theoretic models, one-shot game is a one-round game where players play their best strategies (given the condition of the game) without any chance of changing them afterwards. In fact, players disclose their optimal strategies in one shot. In the road pricing game, one-shot game may never be implemented in practice since it would be the "worst case" scenario. We only consider it in this example for comparison reasons. The cumulative link toll vector resulted (see table table 4.4b) in a system cost of *€375,488*, which is *60%* higher than cooperative outcome *€234,666* of the same game. This reveals that actions of uncoordinated actors can leave the social cost of the network (or the market) far from optimal. The costs of the players are as a result of the cumulative link tolls (link toll total). See the diagonal entries of table 4.3c for an idea of what the cost of a player would be if only this actor operates without other actors. Note that the tolls in *table 4.4* are not unique in general.

Nash equilibrium game

The NE toll pricing game is described to mean a scenario where actors who take part in the game propose tolls in turns in order to satisfy their individual interests. In practice, it can be seen as a parliamentary debate on a tolling scheme, where stakeholders debate on how much tolls to be set, and on which roads and during which hour of the day and so on, all these to the benefit of the individual participating stakeholder or the constituency/industry which he represents. Here actors iteratively solve their individual system problem as given in system (3.22). The game terminates (at NE) when no actor can improve his objective by changing his current toll vector given that other leaders' strategies are fixed. This means that stakeholders agree on a giving tolling scheme and toll pattern if they all perceive it to be fair enough and have all done their best to improve their individual interests. Giving a tolling pattern, if an actor could suggest another pattern that improves his objective without changing other stakeholders' toll strategies, then the pricing game is not yet at Nash equilibrium.

We solve the Nash game using the NIRA-3 [31]. NIRA-3 is a MATLAB package that uses the Nikaido-Isoda function and relaxation algorithm to find unique Nash equilibria in infinite games. An interested reader is referred to Koh [30] for an evolutionary algorithm for EPECs. The toll bound condition of *[0,5]EUR* per link per player helps ensure the existence of Nash equilibrium. Using *alphamethod=0.5,* *precision=[1e-3, 1e-3], and TolCon=TolFun=TolX=1e-3* (for more on the NIRA-3 see [31]). It took NIRA-3 approximately 2 minutes in 70 iterations to find the NE (see Table 3c). The Nash equilibrium game was conducted on MATLAB version 9 running on a 64-bit Windows 7 machine with 4 GB of RAM.

Table 4.4: Results for Different Kinds of Game Models Studied: "First-Best Pricing"

4.4a: Cooperative game among the actors

	Leaders							Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				14.14	261	a–b–e [i]		0
2				15.00	319	a–b–c–e [ii]		261
3				15.00	420	a–b–c–d–e [iii]		0
4				0.00	261	a–c–e [iv]		319
5				0.00	0	a–c–d–e [v]		0
6				15.00	0	a–d–e [vi]		420
7				11.74	580	Total demand		**1000**
8				13.40	420			
Cost	€ 4,065	€ 142,726	€ 87,875	*Toll revenue* ($\theta^T v$)	*€ 27,217*	System cost	=	**€ 234,666**

4.4b: Link tolls for one-shot game and its network effect

	Leaders							Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1	4.43	3.04	1.34	8.81	497	a–b–e [i]		497
2	5.00	4.00	0.40	9.40	0	a–b–c–e [ii]		0
3	5.00	5.00	0.00	10.00	503	a–b–c–d–e [iii]		0
4	0.00	3.46	1.50	4.96	0	a–c–e [iv]		0
5	0.00	0.00	0.85	0.85	0	a–c–d–e [v]		0
6	5.00	0.10	3.93	9.03	497	a–d–e [vi]		503
7	4.82	5.00	4.80	14.61	0	Total demand		**1000**
8	4.52	4.59	0.00	9.11	503			
Cost	€ 4,437	€ 200,771	€ 170,281	*Toll revenue* ($\theta^T v$)	*€ 18,478*	System cost	=	**€ 375,488**

4.4c: Link tolls for a complete Nash game and its network effect

	Leaders							Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1	3.15	1.44	1.43	6.01	281	a–b–e [i]		0
2	2.41	3.75	3.14	9.30	417	a–b–c–e [ii]		281
3	2.06	3.24	3.74	9.04	302	a–b–c–d–e [iii]		0
4	0.45	1.32	0.91	2.68	281	a–c–e [iv]		417
5	0.99	1.04	1.04	3.06	0	a–c–d–e [v]		0
6	4.91	4.88	4.67	14.45	0	a–d–e [vi]		302
7	0.64	2.55	2.24	5.43	698	Total demand		**1000**
8	3.63	3.76	3.95	11.34	302			
Cost	€ 6,353	€ 163,925	€ 100,768	*Toll revenue* ($\theta^T v$)	*€ 16,265*	System cost	=	**€ 271,047**

4.4d: Game with users' interest (in form of minimizing link tolls) represented in the upper level (w.r.t. Nash game - 4.4c)

		Leaders						Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				0.00	281	a–b–e [i]		0
2				0.61	417	a–b–c–e [ii]		281
3				1.60	302	a–b–c–d–e [iii]		0
4				0.00	281	a–c–e [iv]		417
5				0.00	0	a–c–d–e [v]		0
6				6.35	0	a–d–e [vi]		302
7				0.00	698	Total demand		**1000**
8				4.66	302			
Cost	€ 6,353	€ 163,925	€ 100,768	*Toll revenue* ($\theta^T v$)	*€ 2,148*	System cost	=	**€ 271,047**

4.4e: Game with users' interest (in form of minimizing link tolls) represented in the upper level (w.r.t. Cooperative game - 4.4a)

		Leaders						Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				0.00	261	a–b–e [i]		0
2				0.86	319	a–b–c–e [ii]		261
3				0.66	420	a–b–c–d–e [iii]		0
4				0.00	261	a–c–e [iv]		319
5				1.48	0	a–c–d–e [v]		0
6				3.26	0	a–d–e [vi]		420
7				0.00	580	Total demand		**1000**
8				1.86	420			
Cost	€ 4,065	€ 142,726	€ 87,875	*Toll revenue* ($\theta^T v$)	*€ 1,336*	System cost	=	**€ 234,666**

The *Nash game* (Table 3c) shows improvement of *€104,442 (27%)* on the system cost with regard to the *one-shot game*. The iterative process of the *Nash game* tends to inform the actors about other actors' objectives, leading to a sort of "coordinated" game. In some limited sense, actors, during the iterative process, indirectly solve a multi-objective problem [3, 38]. See also that the *cooperative game* improves the social cost of the *Nash game* by *€36,381 (13%)* showing that actions of non-cooperative players may have a negative effect on the system as a whole.

Users interest

Even with its rich potentials to alleviating most of our traffic worries, road pricing has suffered setbacks due to poor acceptance not only from the stakeholders, but also from road users. Road users, even before the implementation of road pricing, perceive that the pricing will take money out of their pockets to the extent that it affects their income. With this in mind, they kick against the implementation of road pricing. To check for this, as earlier mentioned, our models allow users' interest be represented by a stakeholder on the upper level of decision. The objective of this "*user-stakeholder*" is to minimize the total network toll, or to keep link tolls as low (fair) as possible. This stakeholder will play the stakeholders' Nash game described in section 3.1 (and subsection 3.2.2). Alternatively, as mentioned in section 3.1, the "*user-stakeholder*" may allow other stakeholders play the road pricing game, and after the game has reached equilibrium, he then seeks for an alternative (lower) toll vector θ that achieves the same Nash flow pattern for the stakeholders (using the linear system (3.20) of *corollary* 2 - see also Eq.(4.6) below), thus assuring each actor his Nash outcome (see table 4.4d). Table 4.4d shows that each actor can still be assured of his Nash outcome, but with a lower link and total network toll. The total network toll is reduced by *€14,116 (87%)* with respect to the Nash game result (table 4.4c). Furthermore, in the cooperative game, *user-stakeholder* can achieve a total network toll reduction of *€25,880 (95%)* for the road users, and still guarantee other stakeholders their respective entitlement in the grand coalition game (see table 4.4e). For elastic demand, users can gain much in toll reduction by slightly deteriorating the actors' utilities, because for elastic demand, the total toll revenue is the same for all toll patterns (see Eq.(2.18))[79]. This slight deterioration is easily covered by the gain in toll reduction so that the actors are not left worse off than in the Nash game (or in the grand coalition).

$$\boxed{\begin{aligned} &\sum_{a\in A}\left(\beta t_a(\bar{v}_a)+\theta_a\right)\delta_{ar}\geq\lambda_w && \forall r\in R_w,\forall w\in W\\ &\sum_{a\in A}\left(\beta t_a(\bar{v}_a)+\theta_a\right)\bar{v}_a=\sum_{w\in W}\lambda_w\bar{d}_w && \end{aligned}} \tag{4.6}$$

where $\bar{v}$ is the Nash equilibrium link flow vector and λ is the free scalar vector representing the minimum travel costs for the ODs (recall the assertion in Eq.(4.1)). θ is the variable vector of link tolls.

Second-best pricing scheme

So far, we have assumed the possibility of tolling all links, but in practice, such an assumption may not be feasible. For this, economists proposed a tolling scheme that only tolls a part of the network or a subset of the entire network. This scheme may be practically feasible but may lead to a suboptimal network flow pattern, hence the name, second-best road pricing scheme.

To demonstrate the game under the second-best pricing scheme, we assume that links *3* and *6* are not toll-able. In this case, the game in program (3.22) now includes the extra constraint $\theta_3^k = \theta_6^k = 0, \forall k$. Without changing the models and parameters, except for the extra toll constraint, we have the results of the second-best road pricing game displayed in 4.5:
As expected, with the extra condition (on toll), the feasible region decreases (compared with table 4.4). The actors' ideal tolls for the one-shot non-cooperative game lead to a possible "worst case" system cost of *€431,227*. The grand coalition game (table 4.5a) improves the Nash game (table 4.5c) by *€6,897* (*3%*).

Again, "*user-stakeholder*" can achieve a total network toll reduction of *€9,204* (*84%*) with respect to Nash game (see table 4.5d), and *€11,030* (*89%*) with respect to the grand coalition game (see table 4.5e).

Though Nash game improves the system cost of one-shot game, cooperative game is again the overall best in terms of system cost.

Compromise between Nash equilibrium and Multi-objective optimization

Note that it is a mere coincidence for this concrete example game that every player is better off in the grand coalition (table 4.4a and table 4.5a) than in the complete Nash game (table 4.4c and table 4.5c). In general, the system cost is always minimal in the grand coalition game, but there is no guarantee that each player will be better off in the grand coalition in terms of individual cost.

Table 4.5: Results for Different Kinds of Game Models Studied: Second-Best Pricing

4.5a: Cooperative game among the actors

		Leaders						Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				15.00	0	a–b–e [i]		0
2				5.51	569	a–b–c–e [ii]		0
3				0.00	431	a–b–c–d–e [iii]		0
4				0.00	0	a–c–e [iv]		569
5				0.00	0	a–c–d–e [v]		0
6				0.00	0	a–d–e [vi]		431
7				5.51	569	Total demand		**1000**
8				14.14	431			
Cost	€ 4,744	€ 131,613	€ 123,974	*Toll revenue*($\theta^T v$)	*€ 12,374*	System cost	=	**€ 260,331**

4.5b: Link tolls for one-shot game and its network effect

		Leaders						Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1	4.91	3.65	5.00	13.55	302	a–b–e [i]		302
2	5.00	4.04	2.44	11.48	80	a–b–c–e [ii]		0
3	0.00	0.00	0.00	0.00	617	a–b–c–d–e [iii]		0
4	0.00	0.00	0.00	0.00	0	a–c–e [iv]		80
5	0.00	0.00	0.00	0.00	0	a–c–d–e [v]		0
6	0.00	0.00	0.00	0.00	302	a–d–e [vi]		617
7	0.30	0.23	2.44	2.96	80	Total demand		**1000**
8	5.00	5.00	0.00	10.00	617			
Cost	€ 4,644	€ 293,390	€ 133,193	*Toll revenue*($\theta^T v$)	*€ 11,431*	System cost	=	**€ 431,227**

4.5c: Link tolls for a complete Nash game and its network effect

		Leaders						Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1	4.88	5.00	4.33	14.21	0	a–b–e [i]		0
2	0.00	3.94	0.00	3.94	599	a–b–c–e [ii]		0
3	0.00	0.00	0.00	0.00	401	a–b–c–d–e [iii]		0
4	0.05	0.05	0.05	0.15	0	a–c–e [iv]		599
5	0.02	0.02	0.02	0.07	0	a–c–d–e [v]		0
6	0.00	0.00	0.00	0.00	0	a–d–e [vi]		401
7	0.00	5.00	0.27	5.27	599	Total demand		**1000**
8	5.00	4.09	4.51	13.61	401			
Cost	€ 5,363	€ 133,813	€ 128,052	*Toll revenue*($\theta^T v$)	*€ 10,968*	System cost	=	**€ 267,227**

4.5d: Game with users' interest (in form of minimizing link tolls) represented in the upper level (w.r.t. Nash game - 4.5c)

	Leaders							Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				5.01	0	a–b–e [i]		0
2				0.00	599	a–b–c–e [ii]		0
3				0.00	401	a–b–c–d–e [iii]		0
4				0.00	0	a–c–e [iv]		599
5				0.00	0	a–c–d–e [v]		0
6				0.00	0	a–d–e [vi]		401
7				0.00	599	Total demand		**1000**
8				4.40	401			
Cost	€ 5,363	€ 133,813	€ 128,052	*Toll revenue*($\theta^T v$)	*€ 1,764*	System cost	=	**€ 267,227**

4.5e: Game with users' interest (in form of minimizing link tolls) represented in the upper level (w.r.t. Cooperative game - 4.5a)

	Leaders							Path flows
Link	"t"	"e"	"s"	Link toll	Link flow	Paths [j]		[f_j]
1				3.54	0	a–b–e [i]		0
2				0.00	569	a–b–c–e [ii]		0
3				1.56	431	a–b–c–d–e [iii]		0
4				0.00	0	a–c–e [iv]		569
5				0.00	0	a–c–d–e [v]		0
6				0.43	0	a–d–e [vi]		431
7				0.00	569	Total demand		**1000**
8				1.56	431			
Cost	€ 4,744	€ 131,613	€ 123,974	*Toll revenue*($\theta^T v$)	*€ 1,344*	System cost	=	**€ 260,331**

Non-existence of Nash equilibrium

In this part, with a concrete example we demonstrate the non-existence of Nash equilibrium NE with unbounded tolls described in *Corollary 4* subsection 4.1.3 and in Ohazulike et al. [44]. With this example, we show that if stakeholders are free to choose tolls without bounds, and if demand for each origin-destination is fixed, (that is no matter how high the tolls are, the users will still travel), then the road pricing game described in this Chapter has no Nash equilibrium. This means that at each turn of play by an actor, there is always a feasible toll vector that improves this actor's objective without changing other actors' toll strategies. In fact, each actor can always find a toll pattern that achieves his optimal flow pattern.

We again take that actors have the following objectives: *system travel time cost* ($C^t(v)$), *emission cost* ($C^e(v)$) and *safety cost* ($C^s(v)$), respectively. We denote these actors by "t", "e" and "s" respectively. Suppose actors toll/proposes a toll for the network (Figure 3) in a sequential manner ("t" ==> "e" ==> "s" ==> "t" ==> "e" ==> "s"... and so on), we show that they will always find a toll leading to their ideal or optimal flow (and hence their optimal objective value) in each move (see proof of corollary 2). Let us represent the stakeholders' action by θ_i^k which is the link toll vector proposed by actor k in the i^{th} move, and let

θ_i^{k+} be the corresponding positive toll vector that can achieve the same network flows as θ_i^k. We also denote by $\bar{\theta}^k$ the ideal or the first-best toll computed from Eq.(3.19) or precisely Eq.(4.3) for actor $k \in K$. As in Eq.(4.3) $\bar{\theta}^k$ is given by

$$\bar{\theta}^k = \alpha \nabla C^k(\bar{v}^k) - \beta t(\bar{v}^k) - \sum_{\ell \in K \setminus k} \theta^\ell$$

where $\bar{v}^k$ is the ideal flow vector for player k. The quantity $\nabla C^k(\bar{v}^k)$ is the vector of link cost derivatives for actor k evaluated at actor $k's$ optimal flow vector $\bar{v}^k$. The term $t(\bar{v}^k)$ is the link travel time vector again evaluated at actor k*'s* optimal flow vector $\bar{v}^k$. The following gives the numerical values of $\nabla C^k(\bar{v}^k)$ and $t(\bar{v}^k)$ for the three actors:

$$\begin{aligned}
\nabla C^t(\bar{v}^t) &= (1.31,\ 2.03,\ 1.90,\ 0.72,\ 0.57,\ 2.95,\ 2.23,\ 2.36)^T \\
\nabla C^e(\bar{v}^e) &= (74.65,\ 17.12,\ 129.57,\ 83.70,\ 49.36,\ 96.72,\ 154.25,\ 104.89)^T \\
\nabla C^s(\bar{v}^s) &= (24.00,\ 168.00,\ 25.20,\ 0.02,\ 0.01,\ 270.00,\ 13.50,\ 22.95)^T
\end{aligned}$$

$$\begin{aligned}
\beta t(\bar{v}^t) &= (1.06,\ 1.21,\ 1.22,\ 0.72,\ 0.57,\ 1.48,\ 0.95,\ 1.23)^T \\
\beta t(\bar{v}^e) &= (1.77,\ 1.48,\ 1.05,\ 0.72,\ 0.57,\ 6.69,\ 1.11,\ 0.95)^T \\
\beta t(\bar{v}^s) &= (6.88,\ 1.00,\ 1.05,\ 67.88,\ 0.57,\ 1.11,\ 24.70,\ 0.95)^T
\end{aligned}$$

Let *player* "t" be the first to toll or propose a toll for the network, he thus proposes his first-best toll vector

$$\theta_1^t \ = \ \bar{\theta}^t = 1 \times \nabla C^t(\bar{v}^t) - \beta t(\bar{v}^t) = (0.25,\ 0.82,\ 0.68,\ 0.00,\ 0.00,\ 1.47,\ 1.28,\ 1.13)^T$$

then for player "e" to achieve his ideal flow, he will now propose

$$\begin{aligned}
\theta_1^e &= \bar{\theta}^e - \theta_1^t = (1 \times \nabla C^e(\bar{v}^e) - \beta t(\bar{v}^e)) - \theta_1^t \\
&= (72.63,\ 14.82,\ 127.84,\ 82.98,\ 48.79,\ 88.56,\ 151.85,\ 102.81)^T
\end{aligned}$$

Observe that $\bar{\theta}^e = \theta_1^e + \theta_1^t$ meaning that with θ_1^e *player* "e" will change the traffic flow to his ideal flow during his turn of play given the tolls of other players (in this case *player* "t").

The next move will be *player* "s", and with $(\theta_1^e + \theta_1^t)$ in place, to achieve his ideal flow vector, his optimal response is as follows:

$$\begin{aligned}
\theta_1^s &= \bar{\theta}^s - \left(\theta_1^e + \theta_1^t\right) = (1 \times \nabla C^s(\bar{v}^s) - \beta t(\bar{v}^s)) - \left(\theta_1^e + \theta_1^t\right) \\
&= (-55.76,\ 151.36,\ -104.37,\ -150.85,\ -49.35,\ 178.86,\ -164.33,\ -81.94)^T \\
\theta_1^{s+} &= (10,100 \times \nabla C^s(\bar{v}^s) - \beta t(\bar{v}^s)) - \left(\theta_1^e + \theta_1^t\right) \\
&= (242\,320.24,\ 1\,696\,783.36,\ 254\,390.43,\ 0.63,\ 71.84,\ 2\,726\,908.86, \\
&\qquad 136\,172.17,\ 231\,690.11)^T
\end{aligned}$$

Continuing with the game, next will be *player* "t" again, his ideal flow now is achieved by the following positive toll vector, and so on.

Links	θ_2^{t+}	θ_2^{e+}	$\cdots$
	$(925,550 \times \nabla C^t(\bar{v}^t) - \beta t(\bar{v}^t) - \left(\theta_1^e + \theta_1^{s+}\right))$	$(109,650 \times \nabla C^e(\bar{v}^e) - \beta t(\bar{v}^e) - \left(\theta_2^{t+} + \theta_1^{s+}\right))$	$\cdots$
1	$970,618.33$	$6,972,310.81$	$\cdots$
2	$179,281.38$	593.54	
3	$1,503,549.47$	$12,449,409.54$	
4	$663,025.03$	$8,514,678.62$	
5	$530,351.28$	$4,881,900.31$	$\ddots$
6	26.74	$7,877,978.82$	
7	$1,927,650.26$	$14,849,688.96$	
8	$1,950,278.09$	$9,319,219.35$	$\cdots$

Of course these high tolls cannot be realized in practice. The aim of this example is to show that without bounds on tolls, the players can always find a positive toll that yields their ideal flow pattern when it is their turn to propose a toll. Note that these tolls are in general not unique, and therefore, with $(\theta_1^e + \theta_1^t)$ in place, player "s" may find ideal positive toll vectors with tolls lower than what we see in θ_1^{s+} above, the same holds for θ_2^{t+} and θ_2^{e+} (see corollary 1). For example, with $(\theta_1^e + \theta_1^t)$ in place, $\theta_1^{s+} = (22.68,\ 319.65,\ 157.68,\ 82.98,\ 48.79,\ 331.65,\ 11.11,\ 133.10)^T$ will also lead to the ideal flow vector $\bar{v}^s$ for player "s". In fact, the link tolls in θ_1^t and θ_1^e could be made lower than they appear in this example using Eq.(3.20) of *Corollary 2*, and this will lead to even lower tolls in θ_1^{s+} and so on.
Thus, this example is simply to illustrate our statement in *Corollary 3* and *4* that Nash equilibrium does not exist with unbounded tolls. For bounded tolls, the non-existence is demonstrated with the two-node example in section 4.2.

4.4 Nash equilibrium and cooperative game

In the preceding sections (sections 4.1 and 4.2), we have shown that even under the restrictions on tolls (such as $\theta \leq \alpha$, where $\alpha \in \mathbb{R}$) that the existence of Nash equilibrium is not guaranteed. In this section, therefore, we study the necessary conditions for "optimal" solutions of both the Nash equilibrium and the cooperative games. In particular, we analyse the stationary points of the two games and draw remarkable comparisons of the two.

4.4.1 Stationary points of cooperative and non-cooperative game

Nash Equilibrium/Non-cooperative Game Problem

Let tolls be bounded (i.e. that link tolls can not be infinitely large) and suppose that a Nash equilibrium exists, then it will be interesting to know how far the Nash flow vector deviates from the "optimal flow" vector resulting from a grand coalition game or MO solution of system (4.5) or (4.8). In other words, we are interested in knowing how far the competition among the actors can worsen the optimal system cost. Let us consider the fixed demand case.

From Eqs.(3.22) and (4.1) we know that each stakeholder $k \in K$ solves the following problem:

$$\min_{v,\theta^k} Z^k = C^k(v)$$

$$s.t$$

$$\begin{array}{lcll} \Lambda^T\left(\beta t(v)+\theta^k+\sum\limits_{j \in K\backslash k}\tilde{\theta}^j\right) & \geq & \Gamma^T\lambda & \left[\vartheta^k\right] \\ \left(\beta t(v)+\theta^k+\sum\limits_{j \in K\backslash k}\tilde{\theta}^j\right)^T v & = & \tilde{d}^T\lambda & \left[\eta^k\right] \end{array} \quad and \quad \begin{array}{rcll} v & = & \Lambda f & [\psi] \\ \Gamma f & = & \tilde{d} & [\varsigma] \\ f & \geq & 0 & [\rho] \\ (\theta^k & \geq & 0) & \left[\sigma^k\right] \end{array} \tag{4.7}$$

Recall from section 4.1.1 that the link flows (and thus the minimum path cost λ) are not actor dependent.

Remark

Note that due to the non-uniqueness of path flows, it is possible to have different path flows for different actor, but then, without loss of generality, we take one path flow pattern f for all actors and omit the superscript k on f.

The Greek letters $\left(\vartheta^k, \eta^k, \psi, \varsigma, \rho, \sigma^k\right)$ are *KKT multipliers* associated with the constraints. The tilde "$\sim$" indicates fixed parameters in the above optimization problem. System (4.7) involving all players is called an *equilibrium problem subject to user equilibrium condition*, see also [24] for an analysis of such a game.

Now, suppose for every leader k, L_k is the Lagrangian and that the vector $\left(\check{v}, \check{\theta}^k\right)$ solves (4.7) at (Nash) equilibrium, then with *Assumption 1*, there exist multipliers $\left(\check{\vartheta}^k, \check{\eta}^k, \check{\psi}, \check{\varsigma}, \check{\rho}, \check{\sigma}^k\right)$ such that the following *KKT conditions* hold $\forall k \in K$:

KKT 4.7

$$\begin{aligned} L_k \quad = \quad & C^k(v) + \left[\Gamma^T\lambda - \Lambda^T\left(\beta t(v)+\theta^k+\sum_{j \in K\backslash k}\tilde{\theta}^j\right)\right]^T \check{\vartheta}^k + \left[\left(\beta t(v)+\theta^k+\sum_{j\in K\backslash k}\tilde{\theta}^j\right)^T v - \left(\tilde{d}\right)^T(\lambda)\right]\check{\eta}^k \\ & + (\Lambda f - v)^T\check{\psi} + \left(\tilde{d}-\Gamma f\right)^T\check{\varsigma} - f^T\check{\rho} - \left(\theta^k\right)^T\check{\sigma}^k \end{aligned}$$

$$\begin{aligned} \nabla_v L_k &= \nabla C^k(\check{v}) - \beta\left(\Lambda^T\nabla t(\check{v})\right)^T\check{\vartheta}^k + \left(\beta t(\check{v})+\check{\theta}^k+\sum_{j\in K\backslash k}\tilde{\theta}^j+\beta\check{v}^T\nabla t(\check{v})\right)\check{\eta}^k - \check{\psi} = 0 \\ \nabla_f L_k &= \Lambda^T\check{\psi} - \Gamma^T\check{\varsigma} - \check{\rho} = 0 \\ \nabla_{\theta^k} L_k &= -\Lambda\check{\vartheta}^k + \check{v}\check{\eta}^k - \check{\sigma}^k = 0 \\ f^T\rho &= 0 \\ \left(\check{\theta}^k\right)^T\check{\sigma}^k &= 0 \\ \check{\vartheta}^k, \check{\rho}, \check{\sigma}^k &\geq 0 \ \forall k \in K; \ \left[\Gamma^T\lambda - \Lambda^T\left(\beta t(\check{v})+\check{\theta}^k+\sum_{j\in K\backslash k}\tilde{\theta}^j\right)\right]^T\check{\vartheta}^k = 0 \end{aligned}$$

Grand Coalition or Cooperative Game Problem

The grand coalition (GC) game with a toll vector $\theta = \sum_{k\in K} \theta^k$ (assuming that GC assigns θ^k to each actor $k \in K$) is formulated as follows:

$$MO: \quad \min_{v,\theta^k} Z = \sum_{k\in K} C^k(v)$$

$$s.t$$

$$\begin{array}{lcll} \Lambda^T\left(\beta t(v) + \sum_{k\in K}\theta^k\right) & \geq & \Gamma^T\lambda & [\vartheta] \\ \left(\beta t(v) + \sum_{k\in K}\theta^k\right)^T v & = & \left(\tilde{d}\right)^T\lambda & [\eta] \end{array} \quad and \quad \begin{array}{rcll} v & = & \Lambda f & [\psi] \\ \Gamma f & = & \tilde{d} & [\varsigma] \\ f & \geq & 0 & [\rho] \\ (\theta^k & \geq & 0) & [\sigma^k] \quad \forall k \in K \end{array} \tag{4.8}$$

Remark

- The grand coalition game in system (4.8) minimizes the entire system cost, and thus, resulting in “Pareto” optimal system flow $\bar{v}$.
- In (4.8), instead of $\sum_{k\in K}\theta^k$ we could just write θ, the toll θ is written the form of $\sum_{k\in K}\theta^k$ to facilitate the proof of *Corollary 5* below.
- Since systems (4.7) and (4.8) have a non-linear constraint respectively, the efficient way to solve the systems so that we reach the “Pareto” optimum is to:
    1. Solve the convex system for a system optimal flow $\bar{v}$ by omitting the first set of constraints (*EqC_FD*) and the last (the toll) constraint in systems (4.7) and (4.8).
    2. Then, fixing the optimal flow $\bar{v}$, we search for a feasible toll vector $\bar{\theta}$ that satisfies the omitted constraints in step 1. Observe that without *EqC-FD* systems (4.7) and (4.8) are linear. This is the same as solving the linear system (4.6) together with non-negativity of the tolls.
- The solution steps above apply to both first and second-best pricing schemes.

Now, suppose L is the Lagrangian and that $\bar{v}$ and $\bar{\theta}^k$ $\forall k \in K$ solves the grand coalition game (4.8), then with *Assumption 1*, there exist multipliers $\left(\bar{\vartheta}, \bar{\eta}, \bar{\psi}, \bar{\varsigma}, \bar{\rho}, \bar{\sigma}^k\right)$ such that the following *KKT conditions* hold:

KKT 4.8

$$\begin{aligned} L \;=\; & \sum_{k\in K} C^k(v) + \left[\Gamma^T\lambda - \Lambda^T\left(\beta t(v) + \sum_{k\in K}\theta^k\right)\right]^T \bar{\vartheta} + \left[\left(\beta t(v) + \sum_{k\in K}\theta^k\right)^T v - \left(\tilde{d}\right)^T(\lambda)\right]\bar{\eta} \\ & + (\Lambda f - v)^T\bar{\psi} + \left(\tilde{d} - \Gamma f\right)^T\bar{\varsigma} - f^T\bar{\rho} - \left(\theta^k\right)^T\bar{\sigma}^k \end{aligned}$$

$$\begin{array}{rcl} \nabla_v L & = & \sum_{k\in K}\nabla C^k(\bar{v}) - \beta\left(\Lambda^T\nabla t(\bar{v})\right)^T\bar{\vartheta} + \left(\beta t(\bar{v}) + \sum_{k\in K}\bar{\theta}^k + \beta\bar{v}^T\nabla t(\bar{v})\right)\bar{\eta} - \bar{\psi} = 0 \\ \nabla_f L & = & \Lambda^T\bar{\psi} - \Gamma^T\bar{\varsigma} - \bar{\rho} = 0 \\ \nabla_{\theta^k} L & = & -\Lambda\bar{\vartheta} + \bar{v}\bar{\eta} - \bar{\sigma}^k = 0 \\ f^T\bar{\rho} & = & 0, \qquad \left(\bar{\theta}^k\right)\bar{\sigma}^k = 0 \qquad \forall k \in K \\ \bar{\vartheta}, \bar{\rho}, \bar{\sigma}^k & \geq & 0 \;\forall k \in K;, \quad \left[\Gamma^T\lambda - \Lambda^T\left(\beta t(\bar{v}) + \sum_{k\in K}\bar{\theta}^k\right)\right]\bar{\vartheta} = 0 \end{array}$$

If tolls are bounded and suppose that Nash equilibrium exists, then, (theoretically) the (stationary point) solution to the Nash game converges to a stationary point of the cooperative game. We therefore state the following corollary:

Corollary 5. *Assuming that Nash equilibrium exists, then there exist multipliers* $\left(\check{\vartheta}^k, \check{\eta}^k, \check{\psi}, \check{\varsigma}, \check{\rho}, \check{\sigma}^k\right)$ *such that KKT 4.7 holds for all k at (Nash) equilibrium. Moreover, the corresponding (stationary) vector* $\left(\check{v}, \check{\theta}\right)$ *that solves the Nash game (3.23) or (4.7) is also a stationary (possibly a local or global solution) for the grand coalition (GC) game (4.8), where* $\check{\theta} \in R^{|K|}$.

Proof

Since $\left(\check{\vartheta}^k, \check{\eta}^k, \check{\psi}, \check{\varsigma}, \check{\rho}, \check{\sigma}^k\right)$ exists, then, there exists $(\bar{\vartheta}, \bar{\eta}, \bar{\psi}, \bar{\varsigma}, \bar{\rho}, \bar{\sigma}^k) = \sum_{k \epsilon K} \left(\check{\vartheta}^k, \check{\eta}^k, \check{\psi}, \check{\varsigma}, \check{\rho}, \check{\sigma}^k\right)$ such that the corresponding vector$\left(\check{v}, \check{\theta}\right)$ of system (4.7) solves KKT 4.8. For instance, see from KKT 4.8 that

$$
\begin{aligned}
\nabla_v L &= \sum_{k \in K} \nabla C^k(\bar{v}) - \beta \left(\Lambda^T \nabla t(\bar{v})\right)^T \bar{\vartheta} + \left(\beta t(\bar{v}) + \sum_{k \in K} \bar{\theta}^k + \beta \bar{v}^T \nabla t(\bar{v})\right) \bar{\eta} - \bar{\psi} \\
&= \sum_{k \in K} \nabla C^k(\bar{v}) - \beta \left(\Lambda^T \nabla t(\bar{v})\right)^T \sum_{k \epsilon K} \check{\vartheta}^k + \left(\beta t(\bar{v}) + \sum_{k \in K} \bar{\theta}^k + \beta \bar{v}^T \nabla t(\bar{v})\right) \sum_{k \in K} \check{\eta}^k - \sum_{k \in K} \check{\psi} \\
&= \sum_{k \in K} \left(\nabla C^k(\check{v}) - \beta \left(\Lambda^T \nabla t(\check{v})\right)^T \check{\vartheta}^k + \left(\beta t(\check{v}) + \sum_{k \in K} \check{\theta}^k + \beta \check{v}^T \nabla t(\check{v})\right) \check{\eta}^k - \check{\psi}\right) \\
&= 0 \qquad (due\, to\, KKT\, 4.7) \qquad \blacksquare
\end{aligned}
$$

Remark

- Observe that the *KKT conditions* of KKT 4.7 and KKT 4.8 are the same.
- Intuitively, if Nash equilibrium was not a local minimum, then at least one stakeholder could improve his objective contradicting the fact that Nash equilibrium is a stable state where no stakeholder could improve his objective.
- *Corollary 5* is comparable to *Proposition 5.5* in [33]. Our case is more general since we do not assume a completely separable system in the sense that each actor's model is a function of other actors' toll vectors.
- Note that *corollary 5* can be extended to a Nash game between any form of coalitions that the stakeholders deem profitable.

4.4.2 Stability of solutions

To combine their forces, actors or stakeholders or regions may form coalitions during the road pricing game. In this subsection, we will investigate which coalitions are possible, and which are *stable*. We restrict the results here to a fixed demand traffic model, the results can easily be extended to elastic demand traffic model.

Let K denote the set of all actors in the road pricing game. Consider a partition $\mathrm{p}_m = \{S_i, \cdots, S_r\}$ of K. Treating each block S_i of p_m as a single player with objective $\sum_{k\in S_i} C^k(\bar{v}(\mathrm{p}_m))$ or utility $-\sum_{k\in S_i} C^k(\bar{v}(\mathrm{p}_m))$, we may define the *utility* $u(S_i, \mathrm{p}_m)$ of a coalition S_i with respect to p_m as follows

$$u(S_i, \mathrm{p}_m) = -\sum_{k\in S_i} C^k(\bar{v}(\mathrm{p}_m)) \tag{4.9}$$

where $\bar{v}(\mathrm{p}_m)$ is the traffic pattern resulting from a Nash equilibrium (assuming this exists and is unique) in the game with r players, representing coalitions $S_i, \cdots, S_r$. $C^k(v(\mathrm{p}_m))$ is the objective function value. Therefore, a coalition S_i may have a different utility $u(S_i, \mathrm{p}_m)$ depending on the partition p_m (containing S_i as a block) of interest. We assume that the objective of coalition S_i is to optimize the collective interest of the actors in the coalition S_i. Given a partition p_m, the coalitions $S_i \in \mathrm{p}_m$ compete with each other, and the game is a Nash equilibrium game between the coalitions $S_i \in \mathrm{p}_m$ with each coalition S_i solving the following cost minimization problem (see also system (4.7)):

$$\begin{gathered}\underset{v,\theta}{min} \sum_{k\in S_i} C^k(v(\mathrm{p}_m))\\ s.t\\ flow\, feasibility\, conditions\\ EqC\_FD\end{gathered}$$

Where *EqC_FD* denotes the *equilibrium conditions for elastic demand* (see Eq.(4.7)). Observe that we have $\left(2^{\mathrm{K}}-1\right)$ number of unique set identities i since an empty set ϕ is not a coalition. A coalition S_i may appear in one or more partitions. We denote the one set partition or the grand coalition partition by $\mathrm{p}_K = \{K\}$ where K is the set of all players.

Definition 1. (*"Pessimistic" utility of a coalition*): For an arbitrary non-empty subset $S_i \subseteq K$, let us define

$$u(S_i) = \underset{\mathrm{p}_m,\, S_i\in \mathrm{p}_m}{min} u(S_i, \mathrm{p}_m)$$

the "*pessimistic*" utility of S_i, defined by the *worst coalition structure* on $K\backslash S_i$.

Definition 2. (*Stability of a partition*): We say that a partition $\mathrm{p}_m = \{S_i, \cdots, S_r\}$ of K is *stable* if there exist *allocation rules* $x : u(S_i, \mathrm{p}_m) \to \mathbb{R}^{\left|\mathrm{S_i}\right|}$ among all sets $S_i \in \mathrm{p}_m$ such that the utility x_k of every individual player k (in p_m) is at least his *worst case stand-alone utility* $u(\{k\})$. In other words,

$$\mathrm{p}_m = \{S_i, \cdots, S_r\} \text{ is stable} \Longleftrightarrow \exists\, x \in \mathbb{R}^{|\mathrm{K}|} : \begin{array}{r} x(S_i) = u(S_i, \mathrm{p}_m) \\ x_k \geq u(\{k\}) \quad \forall k \end{array}$$

where $\sum_{k\in S_i} x_k = x(S_i) = u(S_i, \mathrm{p}_m)\ \forall S_i$.

Pessimistic k will not leave their coalitions provided that $u(S_i, \mathrm{p}_m)$ is appropriately allocated among the members of S_i.

Definition 3. (*Core*): We say that a *core* exists for the road pricing game if for the grand coalition $\mathrm{p}_K = \{K\}$, there exists an *allocation rule* $x : u(K, \mathrm{p}_K) \rightarrow \mathbb{R}^{|\mathrm{K}|}$ in p_K with utility vector $x_k \in \mathbb{R}^{|\mathrm{K}|}$such that this *allocation rule* guarantees every coalition S_i at least its *pessimistic* utility $u(S_i)$. We thus define a *core* as follows

$$core : \left\{ x \in \mathbb{R}^{|\mathrm{K}|} \middle| \begin{array}{l} x(K) = u(K) \\ x(S_i) \geq u(S_i) \quad \forall S \subset K \end{array} \right\}$$

Corollary 6. *By definition, if core $\neq \phi$, then the grand coalition $\mathrm{p}_K = \{K\}$ is stable ($x \in$ core yields the allocation in definition 2).*

Corollary 7. *A necessary and sufficient condition for a partition p_m of K to be stable is that the utility $u(S_i, \mathrm{p}_m)$ is such that the following holds*

$$u(S_i, \mathrm{p}_m) \geq \sum_{k \in S_i} u(\{k\}) \quad \forall S_i \in \mathrm{p}_m$$

Proof

Proof follows from *definition 2*. ■

Corollary 7. *The resulting Nash equilibrium flow vector $\bar{v}(\mathrm{p}_m)$ for any partition set p_m is a stationary (possibly local or global) point $\check{v}$ of the grand coalition program (4.8).*

Proof

The proof follows from *Corollary 5*. ■

4.5 Summary and conclusion

In this chapter, we studied the classical game theoretical solution concepts ranging from Nash solutions and cooperative solutions to the core of the road pricing game. We showed that in general, the road pricing game has no Nash equilibrium (both in pure and mixed strategies) even when tolls are bounded. With bound restrictions on tolls, the game may possess Nash equilibrium. Investigating the stationary points of the solutions, we revealed that a stationary Nash equilibrium point coincides with that of the grand coalition game. We further proved that if side payments are allowed within coalitions in the cooperative game, then a partition is stable if the core is non-empty, and the total utility of any stable partition is the same as that of the grand coalition game. The chapter also contains numerical examples demonstrating the bi-level multi-stakeholder-multi-objective road pricing game described in chapter 3.

Chapter 5

Optimal Nash inducing Mechanism

5.1 Multi-level model

We have shown so far (in *Chapter 4*) that for the multi-leader model in *Chapter 3*, the existence of a NE cannot be guaranteed. In practice, such a phenomenon is not desirable since it makes the whole pricing game unstable. Further, even if Nash equilibrium exists between the actors, the resulting flow may be far from (Pareto) optimal flow. Therefore, the question we would like to answer is: Can we design a tolling game that yields a stable outcome for the actors? In this chapter, we design a mechanism which induces a NE and even more returns the system optimal strategy as the optimal strategy for each actor. For this model, we will assume that there is a "*grand leader* (GL)" who has authority over all other leaders (extending the foregoing models by adding one more uppermost level in figure 3.1). See him as the central (or federal) government. His sole objective is to ensure (Pareto) optimal social welfare of the entire system. Since competition may lead to tolls that deteriorate the social welfare, and since it is not clear if there is a profit sharing rule which leaves the grand coalition as the only stable coalition among the actors (the core of the game), we develop a mechanism that achieves efficient and desirable global outcomes irrespective of what the actors do. This mechanism aligns the objective of each actor with that of the GL. Thus, actors with once conflicting interests, now indirectly pursue common (GL'*s*) interest. To achieve this goal, the mechanism uses a taxing scheme to simultaneously induce a pure NE and cooperative behaviour among actors, and hence, yielding tolls that are optimal for the system (see figure 5.1). It will turn out later that the tax is the marginal cost which an actor imposes on the system by not considering other actors' objectives during his choice of optimal flow pattern.

We assume that the total revenue generated from the taxing scheme just as the tolls (by the stakeholders) are invested back into the system. We also assume that the actors' utility functions are known to the GL. The *tax* can be seen as what an actor pays for the utility he enjoys for taking part in road pricing, which (the *tax*) depends on the flow pattern proposed or chosen by this actor. Recall that for any solution $\bar{v}$ of the models below, we can always choose a first-best pricing toll which ensures that $\bar{v}$ is UE.

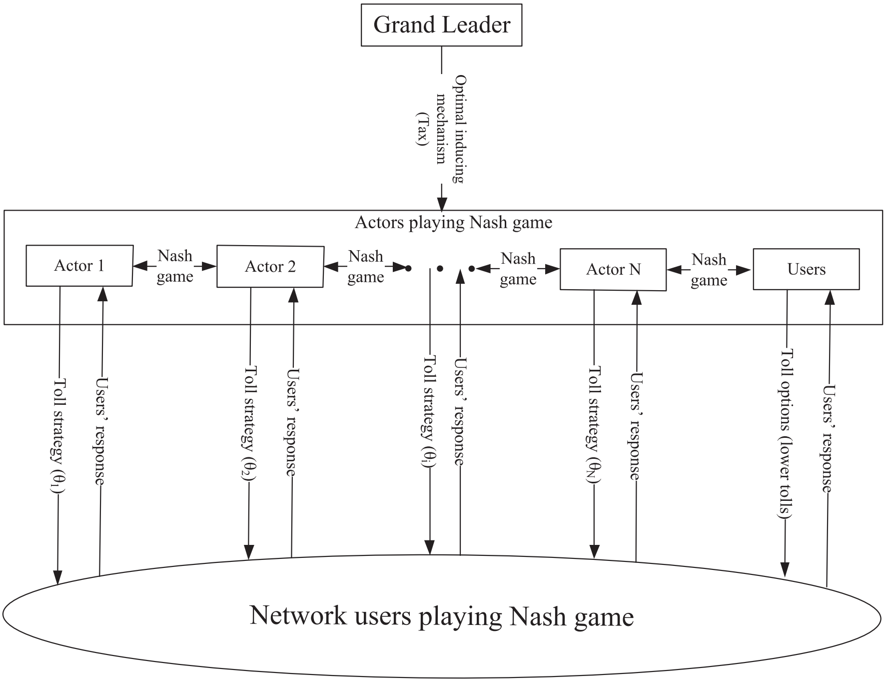


Figure 5.1: Multi-level-Multi-leader road pricing game

5.2 Mathematical formulation of the mechanism

5.2.1 Grand leader's problem

Again, for simplicity, we restrict ourselves to fixed origin-destination demands (extension to Elastic demand model is straightforward). Recall that for any solution link flow vector $\bar{v}$, we can always choose a first-best pricing toll which ensures that $\bar{v}$ is UE using Eq.(4.6) for example. We therefore state the objective of the grand leader to search for an optimal flow pattern $\bar{v}$.

The *GL* problem is a multi-objective (grand coalition) optimization problem that searches for a flow pattern minimizing the entire system cost. Using the weighted sum method (see Eq.(2.22)), we aggregate the objectives into one, converting it to a single objective optimization. Note that we have used equal weights on the objectives. Furthermore, note that the grand leader "reserves the right" to choose weights on the objectives as he deems socially equitable/profitable for the system. The formulation is as follows (see also Eq.2.23):

$$\min_{v} Z(v) = \sum_{k \in K} C^k(v) \quad s.t \quad \begin{array}{rll} v & = \Lambda f & [\psi] \\ \Gamma f & = \bar{d} & [\lambda] \\ f & \geq 0 & [\rho] \end{array} \tag{5.1}$$

The constraints are the flow feasibility constraints, and $\psi \in \mathbb{R}^{|A|}, \lambda \in \mathbb{R}^{|W|}, \rho \in \mathbb{R}^{|R|}$ are the *KKT* multipliers associated with the constraints.

Let L be the Lagrangian and $\bar{v}$ the solution to *(5.1)*, then with *Assumption 1*, there exists $(\bar{\psi}, \bar{\lambda}, \bar{\rho})$ such that the following *KKT* optimality conditions hold:

$$
\begin{aligned}
L &= \sum_{k \in K} C^k(v) + (\Lambda f - v)^T \psi + (\bar{d} - \Gamma f)^T \lambda - f^T \rho \\
\nabla_v L &= \sum_{k \in K} \nabla C_k(\bar{v}) - \bar{\psi} = 0 \;\; or \;\; \frac{d}{dv_a} \sum_{k \in K} C^k_a(\bar{v}_a) - \bar{\psi}_a = 0 \qquad \forall a \in A \qquad (5.2) \\
\nabla_f L &= \Lambda^T \bar{\psi} - \Gamma^T \bar{\lambda} - \bar{\rho} = 0 \;\; or \;\; \sum_{a \in A} \bar{\psi}_a \delta_{ar} - \bar{\lambda}_w - \bar{\rho}_r = 0 \quad \forall r \in R_w, \forall w \in W (5.3) \\
f^T \bar{\rho} &= 0 \;\; or \;\; \bar{\rho}_r f_r = 0 \qquad \forall r \in R \qquad (5.4) \\
\bar{\rho} &\geq 0, \; f \geq 0
\end{aligned}
$$

Eq.(5.4) is called complementarity equation. Again δ_{ar} is a binary variable that equals *1* if link a belongs to path r, and *0* otherwise.

5.2.2 Stakeholders' (or Actor's) problem

Having shown that NE does not exist in general, we discuss a mechanism where the GL chooses appropriate taxes x^k, $k \in K$ which force the game into a NE. This taxing mechanism is as follows:

The GL penalizes (taxes) the k^{th} actor by $v^T x^k$, where v^T is the transpose vector of link flows, and $x^k \in \mathbb{R}^{|A|}$ is a leader specific constant tax vector. The *tax* $\left(v^k\right)^T x^k$ should be seen as the marginal cost which actor $k \in K$ imposes on the system by not considering other actors' objectives during his choice of flow vector v. Henceforth we will omit the superscript k on the flow vector v^k of actor k due to Eq.(4.1).

Now for *fixed tax* x^k each of the stakeholders $k \in K$ solves the following optimization problem:

$$
\min_v Z^k(v) = C^k(v) + v^T x^k \quad s.t \quad
\begin{array}{cc}
v = \Lambda f & \psi \\
\Gamma f = \bar{d} & \lambda \\
f \geq 0 & \rho
\end{array}
\qquad (5.5)
$$

Let L be the Lagrangian and $\tilde{v}$ the solution to (5.5), then, with (ψ, λ, ρ), the following *KKT* conditions hold:

$$
\begin{aligned}
L &= C^k(v) + v^T x^k + (\Lambda f - v)^T \psi + (\bar{d} - \Gamma f)^T \lambda + - f^T \rho \\
\nabla_v L &= \nabla C^k(\tilde{v}) + x^k - \psi = 0 \;\; or \;\; \frac{d}{dv_a} C^k_a(\tilde{v}_a) + x^k_a - \psi_a = 0 \qquad \forall a \in A \qquad (5.6) \\
\nabla_f L &= \Lambda^T \psi - \Gamma^T \lambda - \rho = 0 \;\; or \;\; \sum_{a \in A} \psi_a \delta_{ar} - \lambda_w - \rho_r = 0 \quad \forall r \in R_w, \forall w \in W (5.7) \\
f^T \rho &= 0 \;\; or \;\; \rho_r f_r = 0 \qquad \forall r \in R \qquad (5.8) \\
\rho &\geq 0, \; f \geq 0
\end{aligned}
$$

Observe that the only difference between the *GL's* and the *stakeholder's KKT* conditions is in Eqs.(5.2) and (5.6). Now, the *GL* can choose *taxes* x^k $\forall k \in K$

such that the actors' optimal strategies coincide with the optimal strategy $\bar{v}$ of the *GL*. We define *the optimal strategy* $\bar{v}$ *of the GL* to be the solution to the *GL's* problem (5.1). To force Eq.(5.6) to be exactly the same as Eq.(5.2), *i.e*

$$\begin{aligned} \nabla C^k(v)\Big|_{v=\tilde{v}} + x^k - \psi = \sum_{k\in K} \nabla C^k(v)\Bigg|_{v=\bar{v}} - \bar{\psi} \\ x^k = \sum_{k\in K} \nabla C^k(v)\Bigg|_{v=\bar{v}} - \nabla C^k(v)\Big|_{v=\tilde{v}} + \psi - \bar{\psi} \end{aligned}$$

To achieve this, for each k we can choose the same flow $v^k = \tilde{v} = \bar{v}$ and $\psi = \bar{\psi}$, and choose the taxes

$$\boxed{x^k = \sum_{l\in K\backslash k} \nabla C^l(v)\Bigg|_{\bar{v}}} \tag{5.9}$$

Note that by *Assumption 1*, the convexity assumptions on $C^k(v)$ ensure that the solutions $\bar{v}$ and $\tilde{v}$ to programs (5.1) and (5.5) respectively, are unique.

To summarize:

- By our construction, we have shown that if the *GL* chooses *taxes* $x^k\ \forall k \in K$ as in (5.9) then the solution strategies $\tilde{v}$ (or the Nash equilibrium outcome of problem (5.5)) of the all stakeholders in (5.5) coincide with *GL* solution $\bar{v}$ in (5.1).
- So, any toll vector $\bar{\theta}$ which induces $\bar{v}$ to be a UE can be chosen (e.g. the first-best toll of the form given in (3.19)) by the first actor (say actor k). Since the flow $v^k = \tilde{v} = \bar{v}$ is optimal for all actors, it then means that together with the taxes x^k the toll $\bar{\theta}$ is also optimal for other actors, and therefore, is a cumulative NE toll in the Nash game of section 3.2.2. In fact, one possible NE toll vectors for the players in the Nash game of section 3.2.2 can be chosen as follows: $\tilde{\theta}^k = \bar{\theta} = \nabla \sum_k C^k(\bar{v}) - \beta t(\bar{v})$ (first-best toll of the form given in (3.19)) and $\tilde{\theta}^l = 0\ \forall l \in K\backslash k$ assuming that actor k makes the first move (i.e. actor k is *player* 1).

Remark: Observe from Eq.(5.6) that a taxing scheme defined by the tax function $v^T x^k$ with

$$\boxed{x^k = \bar{\psi} - \nabla C^k(v)\Big|_{v=\bar{v}}} \tag{5.10}$$

where $\bar{\psi}$ is as defined in *grand leader's* problem, is also an optimal Nash inducing scheme. We call Eqs.(5.9) & (5.10) the *first-best taxes*.

This means that with the taxes in (5.9) or (5.10), the objectives of the players are now aligned, and they now pursue common interests. This mechanism is analogous to the first-best pricing where a stakeholder, knowing the road users' reaction (user equilibrium), chooses a toll such that the user equilibrium coincides with his desired flow pattern. So Eq.(5.9) could be called *first-best pricing taxes*.

Interpretation of the taxes: Now we interpret the tax function $v^T x^k$ for actor $k \in K$. The term $\nabla C^l(v)$ in Eq.(5.9) measures how sensitive actor l's objective is to changes in the link flow vector v. A high value of $\nabla C^l(v)$ means that the objective $C_l(v)$ of actor l is very sensitive to changes in link volumes

v, and a low value suggests otherwise. The whole term $\sum_{l \in K \backslash k} \nabla C^l(v)$ in (5.9) measures the cumulative change in the objectives (of actors $l \in K\backslash k$) with respective to a unit increase in the link flow v. Consequently, the tax function $v^T x^k = v^T \left(\sum_{l \in K \backslash k} \nabla C^l(v)\right)$ for actor k measures the total change in other actors' objectives ($l \in K\backslash k$) when the link flow vector is increased at v. A large tax $v^T x^k$ on k means that actor *k's* "optimal" choice of v contradicts to a large extent the interests of actors in $K\backslash k$. In fact, the tax function $v^T x^k$ is the marginal cost which actor $k \in K$ imposes on the system by not considering other actors' objectives during his choice of v. Therefore, by taxing actor k the quantity $v^T x^k$, we internalize in his objective, the cost he imposes on other actors, and thereby, indirectly make him aware of other actors' objective. In this way, his choice for the link flow vector v or more precisely, his choice of toll θ^k, inducing v, is optimal for the system.

In general, the objective $Z(v) = C^{GL}(v)$ of the Grand leader could be different from the one stated in Eq.(5.1). Thus, the generalisation of the tax x^k given in Eq.(5.9) for stakeholder k

$$\boxed{x^k = \nabla C^{GL}(v) - \nabla C^k(v)\Big|_{\bar{v}}} \tag{5.11}$$

where $\bar{v}$ is the optimal flow vector for the Grand leader's problem.

The tax x^k in Eq.(5.11) measures the difference between the sensitivity of the *GL's* objective and the actor *k's* objective to changes in the link flow vector v. This means that each actor is taxed based on how sensitive his objective is to the link flow vector v, as compared to the sensitivity of the *GL's* objective to v evaluated at $\bar{v}$. Therefore, actors pay less taxes if there objectives are somewhat aligned with the objective of the *GL*, and more taxes if their objectives differ much from that of the *GL*. Observe that the tax x^k may take a negative value, *this means that actors can actually receive subsidies to play according to the GL's desired flow pattern v. Note that the use of subsidies to steer stakeholders' actions could lead to a corrupt system, where one or some of the stakeholders would lobby the Grand leader or the "central government" to use the taxing mechanism in their favour.*

5.2.3 Users problem

With the objective of the stakeholders' problem of system (4.7) replaced with the taxed objective in system (5.5), stakeholders compete for optimal tolls to optimize their individual objectives while ensuring user equilibrium UE. With the taxed objective, the problem reduces to the usual bi-level toll pricing game. When Nash equilibrium is reached, then with respect to the cumulative Nash toll, road users will route themselves to satisfy Wardrop's or user equilibrium.

5.3 Analysis of the inducing scheme

5.3.1 Flexible taxing scheme

It will be interesting to see if there are other taxing schemes (other than those defined in Eqs.(5.9) and (5.10)) induce NE and system optimal behaviour among the actors. It turns out that as in the first-best tolling mechanism described in subsection 2.1.6, there are (possibly) infinitely many values for x^k in the taxing schemes $v^T x^k$(other than those defined in Eqs.(5.9) and (5.10)) that induce optimal Nash. *We define optimal Nash to mean a Nash equilibrium point that coincides with the* GL's *optimal point.* Using the *KKT optimality conditions* above, we first note the following corollary

Corollary 8. *If $\tilde{v}$ is the optimal flow vector in system (5.5) for actor $k \in K$, then the following holds:*

$$\boxed{\begin{aligned} \sum_{a\in A}\left(\frac{d}{dv_a}C_a^k(v_a)\bigg|_{v_a=\tilde{v}_a} + x_a^k\right)\delta_{ar} &= \lambda_w + \rho_r \geq \lambda_w \qquad \forall r \,\epsilon\, R_w,\ \forall w \,\epsilon\, W \\ \sum_{a\in A}\left(\frac{d}{dv_a}C_a^k(v_a)\bigg|_{v_a=\tilde{v}_a} + x_a^k\right)\tilde{v}_a &= \sum_{w\in W}\lambda_w \bar{d}_w \end{aligned}} \tag{5.12}$$

condensed to

$$\boxed{\begin{aligned} \Lambda^T\left(\nabla C^k(v)\Big|_{v=\tilde{v}} + x^k\right) &\geq \Gamma^T\lambda \\ \left(\nabla C^k(v)\Big|_{v=\tilde{v}} + x^k\right)^T\tilde{v} &= \bar{d}^T\lambda \end{aligned}} \quad \text{for some } \lambda \geq 0. \tag{5.13}$$

Proof

The proof follows the idea of the first-best toll described in *Chapter 3* as well as the proof for the alternative first-best pricing tolls given in [79, 42]. ■

The first line of Eq.(5.12) states that each leader $k \in K$ would want each road user to follow the route that minimizes his (user's) travel cost with respect to his (actor's) objective function. The second line balances the network travel cost (*w.r.t. k's* objective function) The following result on *first-best taxes* is analogous to *Corollaries 1&2* (in subsections 2.1.6 and 3.2.1 respectively).

Corollary 9. *Suppose $\bar{v}$ solves the GL's problem (5.1), then, any taxing scheme $v^T x^k$ such that x^k satisfies the following linear conditions is an optimal Nash inducing taxing scheme on leader $k \in K$:*

$$\boxed{\begin{aligned} \Lambda^T\left(\nabla C^k(v)\Big|_{v=\bar{v}} + x^k\right) &\geq \Gamma^T\lambda \\ \left(\nabla C^k(v)\Big|_{v=\bar{v}} + x^k\right)^T\bar{v} &= \bar{d}^T\lambda \end{aligned}} \quad \text{for some } \lambda \geq 0 \tag{5.14}$$

Proof

The proof follows from *Corollary 8.* ■

Remarks

1. Equations (5.9) and (5.10) directly satisfy condition (5.14).
2. By just knowing the objective $C_k(v)$ of stakeholder *k,* the flexible taxing scheme enables the grand leader (with a desired flow pattern $\bar{v}$) to determine x^k for stakeholder k (see Eq.5.14); In contrary, Equation (5.9) requires that the GL knows other stakeholders' objective, and Equation (5.10) yields only one possible value for x^k. The taxing mechanism can be compared with the usual social taxing scheme where taxes depend on income, and you only need to know one's income to compute the tax.

3. Furthermore, any of the stakeholders can pull out of the road pricing scheme/game without altering the model.

5.3.2 Secondary objectives on the taxing scheme

Equation (5.14) suggests that we can set secondary objectives on these *taxes*. The following has intuitive meaning:

- Keep each actor's tax as low as possible. This can be achieved by solving the following linear system:

$$\min_{x^k} \bar{v}^T x^k \qquad s.t \qquad \begin{array}{c} \Lambda^T \left(\nabla C^k(\bar{v}) + x^k\right) \geq \Gamma^T \lambda \\ \left(\nabla C^k(\bar{v}) + x^k\right)^T \bar{v} = \bar{d}^T \lambda \end{array} \qquad \forall k \,\epsilon\, K \tag{5.15}$$

  where $\bar{v}$ is the GL desired link flow vector. Alternatively, for fairness, the GL may want to levy a *flat tax* on all stakeholders, e.g. $\bar{v}^T x^k = M$, for big enough M.

5.4 Coalition among leaders under the mechanism

In game theory and mechanism design, stability of solutions has always been of great interest. In this section, we would want to investigate how stable the optimal Nash inducing mechanism is. In particular, if side payments are allowed for the actors, we would like to know whether the actors will be better off forming coalitions than staying as a single player in the road pricing game under the taxing scheme described above.

It turns out that the Nash inducing scheme described above is stable. In particular, we prove that there is no coalition formed by actors that will lead to a better pay-off than in the induced Nash scenario. We therefore, state the following:

Lemma 2. *With the taxing scheme described above, there does not exist a coalition in which any of the leaders is better off than in the induced Nash scenario (where each coalition comprises a singleton).*

Proof

Suppose such a coalition exists, say with a feasible flow vector $\hat{v} \neq \bar{v}$ in which actor $k \in K$ is better off than in the induced Nash scenario (where each coalition is comprised of a single actor), then, it simply contradicts the already established fact in subsection 5.2.2 that the induced Nash flow vector $\bar{v} \neq \hat{v}$ is the optimal (idle) flow vector for all leaders under the taxing scheme. Hence, such a coalition does not exist. In fact, for an arbitrary coalition say of two leaders k and m:

Let

$$\tilde{C}^k(v) = C^k(v) + v^T x^k$$
$$\tilde{C}^m(v) = C^m(v) + v^T x^m$$

where

$$x^k = \sum_{l \in K \setminus k} \nabla C^l(v)\Big|_{v=\bar{v}}, \qquad x^m = \sum_{l \in K \setminus m} \nabla C^l(v)\Big|_{v=\bar{v}}$$

as given in Eq.(5.9) and $\bar{v}$ is the GL solution (see (5.1)). After coalition, their objective function is

$$\tilde{C}^k(v) + \tilde{C}^m(v) \;=\; C^k(v) + C^m(v) + v^T(x^k + x^m) \tag{5.16}$$

Given that $\tilde{v}\,\epsilon\, V$ minimizes Eq.(5.16), then, KKT conditions for the minimization problem differ from those of stakeholders' problem (Eq.5.5) only in $\nabla_v L$, which is now given by:

$$\nabla_v L = \nabla C^k(v)\Big|_{v=\tilde{v}} + \nabla C^m(v)|_{v=\tilde{v}} + \sum_{l\in K\backslash k} \nabla C^l(v)\Bigg|_{v=\bar{v}} + \sum_{l\in K\backslash m} \nabla C^l(v)\Bigg|_{v=\bar{v}} - \psi \tag{5.17}$$

Where $\bar{v}$ is the GL's optimal flow pattern. Since $\bar{\psi}$ exists for the GL's problem, then with $\psi = 2\bar{\psi}$, see that $\tilde{v} = \bar{v}$ is a feasible solution for Eq.(5.17), and hence optimal (see Eq.5.2). Therefore, for $\tilde{v} = \bar{v}$, Eq.(5.17) becomes

$$\nabla_v L \;= 2\left(\sum_{l\in K} \nabla C_l(v)\Bigg|_{v=\bar{v}}\right) - 2\bar{\psi} = \;\; 0 \tag{5.18}$$

due to Eq.(5.2), confirming that the GL's optimal flow vector $\bar{v}$ is also optimal for the coalescing stakeholders, k and m. ■

In the taxing scheme described above, we assumed that we can toll all links without bounds. This is the so called first-best pricing scheme. In the next section, we discuss the taxing mechanism with toll constraints/bounds. It is worthwhile stating that when tolls are not allowed on some links (the so called second-best pricing scheme), we face even a harder problem.

5.5 Optimal Nash inducing scheme for second-best pricing

Due to practical flavour of the second-best road pricing scheme, where only a subset of network links is allowed to be tolled, we establish in this subsection results on the second-best scheme for the Nash equilibrium inducing mechanism. In particular, we would want to know how robust our Nash inducing mechanism is when the tolls are constrained.

5.5.1 Unbounded non-negative tolls

Here, we will see that the taxing scheme is also applicable when extra conditions on tolls are present, and the first-best tolls are no longer feasible.

Grand leader's problem

Suppose, we have the toll constraints $\theta_a \geq 0\,\forall a \in A$, and $\theta_a = 0\ \forall a \in Y \subseteq A$. As a single-level non-linear program, the bi-level optimization problem (mathematical problem with equilibrium conditions - MPEC) can be reformulated as follows (see also Eq.2.20):

$$\min_{v,\theta,\lambda} \sum_{k\in K} C^k(v) \qquad s.t \qquad \begin{array}{rcl} \Lambda^T\left(\beta t(v)+\theta\right) & \geq & \Gamma^T\lambda \\ \left(\beta t(v)+\theta\right)^T v & = & \bar{d}^T\lambda \\ \lambda & \geq & 0 \\ \theta_a & \geq & 0 \;\; \forall a\in A \\ \theta_a & = & 0 \;\; \forall a\in Y \\ v & \in & V \end{array} \tag{5.19}$$

The objective minimizes the system cost. The first three constraints are the usual conditions ensuring that resulting flow is in user equilibrium, and the last three ensure that the resulting flows and tolls are feasible (see also Eq.2.1).

Stakeholders' problem

Each actor $k \in K$, instead of Eq.(5.5), now solves the following non-linear program (see system 3.22):

$$\min_{v,\theta^k,\lambda} Z^k(v) = C^k(v) + v^T x^k$$

$$s.t$$

$$\begin{array}{rcll} \Lambda^T\left[\beta t(v) + \left(\theta^k + \sum_{l\in K\backslash k} \bar{\theta}^l\right)\right] & \geq & \Gamma^T\lambda & \\ \left[\beta t(v) + \left(\theta^k + \sum_{l\in K\backslash k} \bar{\theta}^l\right)\right]^T v & = & \bar{d}^T\lambda & \\ \theta_a^k & \geq & 0 & \forall a\in A \\ \theta_a^k & = & 0 & \forall a\in Y \\ v & \in & V & \\ \lambda & \geq & 0 & \end{array} \tag{5.20}$$

If we compare the *KKT conditions* of systems (5.19) and (5.20) (under the assumption that solutions of (5.19) and (5.20) satisfy the KKT conditions), then as in section 5.2, we have the following:

- let $\bar{v}$ be the solution to program (5.19). If the *GL* chooses taxes x^k as in (5.9), then $\bar{v}$ is also optimal for all stakeholders' problems (5.20).

In fact, there is no problem arising from the extra conditions on tolls since system (5.20) holds for all $k \in K$, so the resulting toll vector $\theta = \sum_{k\epsilon K} \theta^k$ will satisfy $\theta_a = 0$, $\forall a \in Y$ (since $\theta_a^k = 0$ $\forall a \in Y$) which means that system (5.20) satisfies/captures the toll constraint in the *GL's* problem (5.19).

Remarks

The *GL's* optimal link toll vector $\bar{\theta}$ is a valid (cumulative) Nash toll vector for the actors (recall the optimal Nash inducing scheme), *i.e.*, $\theta^k = \bar{\theta}$ and $\theta^l = 0$ for $l \neq k$ yields a NE (inducing flow $\bar{v}$).

One possible optimal toll vector for the actors is $\tilde{\theta}^k = \bar{\theta}$ and $\tilde{\theta}^l = 0$ $\forall l \in K\backslash k$ assuming that actor k makes the first move (i.e. actor k is *player* 1). Though these optimal link tolls are not unique in general, *a toll vector* $\tilde{\theta}^k$ $\forall k \in K$ *is Nash optimal for the actors if the cumulative Nash toll* $\sum_{k\in K} \tilde{\theta}_a^k = \theta_a$ $\forall a \in A$ *yields the unique flow vector* $\bar{v}$ *that solves the GL's problem* (5.19).

5.5.2 Bounded tolls

We consider further constraints on the tolls, for example, upper bound constraints requiring that $\theta_a \leq \phi_a$ $\forall a \in A$; with $\phi_a \in \mathbb{R}_+$. In this case, and for equity reasons, one may assume that each stakeholder has a link toll bound given by: $\phi_a^k = \frac{\phi_a}{|K|}$. In fact, we make the following observation:

1. Any link toll vector $\bar{\theta}_a \leq \phi_a$ $\forall a \in A$ that yields the unique flow vector $\bar{v}$ which solves the *GL's* problem (5.19) is also a valid Nash link toll vector for the actors, with
$$\sum_{k\in K} \tilde{\theta}_a^k = \bar{\theta}_a \qquad \forall a \in A \tag{5.21}$$
irrespective of how $\bar{\theta}_a$ is distributed among the actors.
2. The toll vectors $\tilde{\theta}$ in 1 are in general not unique. This means that a link toll outcome of the actors' Nash game may be optimal and at the same time not sum up to the pre-calculated *GL's* toll as in Equation (5.21).
3. As stated in 2, even though these link tolls are not unique in general, but then, *a toll vector* $\tilde{\theta}^k$ $\forall k \in K$ *is Nash optimal for the actors if the cumulative Nash toll* $\sum_{k\in K} \tilde{\theta}_a^k = \theta_a$ $\forall a \in A$ *yields the unique flow vector* $\bar{v}$ *that solves the GL's problem* (5.19).

General application of the optimal Nash equilibrium inducing mechanism

The optimal inducing mechanism can also be used to induce a system optimal performance in the following scenarios:

1. Malicious nodes in car to car communication where cars exchange data/information within a limited time frame.

   In telecommunication networks where cars equipped with sensors, exchange (say) traffic and environmental information (such as weather, road closure, accidents and so on), it is assumed that "rational" cars will send a piece of information depending on what they get in return. This decision is made within a limited time since the cars are in motion and have limited radius of broadcast. This means that car "A" will "only" send valuable data to car "B" if car "A" gets somehow a worthwhile data in return. This type of model, of course, may not be socially optimal, so by using our mechanism, we can induce a system optimal data exchange between the cars by making the system optimal data exchange the optimal strategies for the cars (Schwartz et al. [58, 56, 55, 57]).
2. Local authorities tolling separate regions of the network.

   As we explained in our taxing scheme, the GL now will be the federal government asking each local authority to pay tax to the federal government based on the tolls they collect on these roads. But then, these taxes will be chosen in a way to induce optimal road tolls among these local authorities. The induced optimal tolls are such that the entire nation's network flow is optimized or at least enables the optimization of the GL's objective.

3. Energy producers in the energy market liberalization problem.

   Governments can force energy marketers to set prices that are socially desirable using the taxing scheme model. The government will tax marketers' profit in a way that if they try to maximize their profit, they will end up setting the socially desired price.

4. Agents in the principal-agent model.

   Principal will set tax on agents' income such that agents' optimal salary quotations will be the desired amount that the principal is willing to pay.

5. Internet providers in the providers-subscribers Internet price setting problem. As in 3
6. Competition of firms over the same market shares. As in 3
7. Employees that have flexibility on the number of workdays.

   Employer will set taxes based on the number of working hours in a way that when the employees try to maximize their net income, they will end up working exactly the number of working hours desired by the employer.

5.6 Summary and conclusions

Following our demonstration in chapter 4 that both pure and mixed Nash equilibrium may not exist for the road pricing game, we have developed in this chapter a mechanism that simultaneously induces a pure NE and cooperative behaviour among actors, thus, yielding optimal tolls for the system. In the concluding part of the chapter, we enumerate many other applications that can mimic our optimal mechanism design.

Chapter 6

A comparison of genetic algorithm with our game theoretical approach in solving multi-objective problems

6.1 Introduction

As we already mentioned, road tolling/pricing is a well-accepted technique in transportation economics to combat traffic externalities such as congestion, emission, noise, or safety issues. The problem is how much to toll on which road segment such that traffic is efficiently distributed in a given network. Efficiency here refers to a traffic pattern that optimizes the externalities of interest. Since the mentioned traffic externalities may very well be in conflict with each other, a toll pattern and hence a traffic pattern that optimizes one externality may deteriorate other externalities. Consequently, there is no specific toll pattern that is best for all objectives. For this reason, it may be desirable to list all non-dominated ("*Pareto optimal*") solutions. These solutions can then be presented to the decision or policy makers as possible candidate solution(s). In other words, we want to solve a general multi-objective problem (MOPs) using the game mechanism we described earlier in this thesis. The fact that almost all known (genetic) algorithms for solving MOPs depend on *Pareto dominance* to generate non-dominated solutions makes it so difficult to solve MOPs when the objective number exceeds four. The algorithms begin to deteriorate in efficiency as the objective number increases. The game mechanism we describe does not deteriorate with objective number, and has nothing to do with *Pareto dominance,* so it could be a promising tool for solving multi-objective problems.

Genetic algorithms (GAs) are widely accepted by researchers as a method of solving multi-objective optimization problems, at least for listing a high quality approximation of the Pareto front of an MOP. Many researchers have turned attention to solving multi-objective problems using genetic algorithms (GAs) over the recent years. This is mostly because of their robustness in listing layers of Pareto fronts using the so called Pareto ranking. The interested reader may consult [14] for a general review of the field of GAs in multi-objective optimization and [15] for an extensive description of the field. The articles discuss some of the most representative algorithms that have been developed so far, as well as some of their applications. Methodological issues related to the use of multi-objective evolutionary algorithms, as well as some of the current and future research trends in the area are discussed in [14]. Our motivation for this Chapter stems from the recommendation in [14] to seek for "*alternative mechanisms into an evolutionary algorithm to generate non-dominated solutions without relying on Pareto ranking (e.g., adopting concepts from game theory)*". There have been efforts to incorporate game theory to enhance the performance of GAs. In order to force a GA to list Nash equilibrium points, [59] developed an algorithm that merges GAs and Nash strategy. Application of such merge to domain decomposition method (DDM) - nozzle optimization problems is studied in [48]. Other applications can be found in [13, 53]. GAs have been used to find solutions to some game theoretic problems [26]. In their paper [26], they used a GA to find the optimal strategy of players in a given game. On the other hand, game theorists have incorporated the idea of evolution into game theory in what is now known as an evolutionary game theory. These efforts to merge the two disciplines, however, fail to look at the results separately. Therefore, in this Chapter, we address the following: firstly, we use a game theoretic approach to construct an approximation of the Pareto front of a multi-objective problem, and secondly, we compare this Pareto front with a Pareto front that is constructed by the well-known genetic algorithm, *non-dominated sorting genetic algorithm II* (*NSGA-II*). *NSGA-II* is a widely accepted GA that has been used by researchers and is developed in [16].

The remainder of the Chapter is organized as follows: section 6.2 gives the general overview of traffic externalities and road pricing. Section 6.3 describes the problem, and the solution methods employed: *NSGA-II* and game theoretical approaches. In section 6.4, we demonstrate our models using a numerical example, and finally, section 6.5 concludes the Chapter.

6.2 Traffic externalities and road pricing

Over the past years, vehicle ownership has increased tremendously. It has been realized that the social cost of owning and driving a vehicle does not only include the purchase, fuel, and maintenance fees, but also the cost of man hour loss to congestion and road maintenance, costs of health issues resulting from accidents, exposure to poisonous compounds from car exhaust pipes, and high noise level from vehicles. So, optimizing traffic flow requires a model that optimizes several objectives, which may conflict with each other. Optimization of more than one traffic externality is not a novel idea, but what is novel is that we are using a game theoretical approach to list elements in the solution space. The motivation for solving the multi-objective problem using a game theoretical approach stems from the limitations and critics arising from the traditional way of modelling road pricing. In Chapter 3, we model the road pricing as a multi-leader Stackelberg game where the leaders represent the objectives (or the road traffic externalities) and compete for toll patterns that satisfy their individual interests. At the lower level, are the road users. For a full description of the game, the reader is referred to Chapter 3.

In addition to results in Chapter 3, we would like to investigate in this Chapter if the game theoretic approach can be used to list Pareto solution of a multi-objective optimization problem. And if yes, how would the results from the game compare with the results of the multi-objective optimization using genetic algorithms? In what follows, we provide a brief explanation of the solution methods we have used in this Chapter.

6.2.1 General traffic model and Pareto optimality

Let $G = (N, A)$ be a network, with N the set of all nodes and A the set of (directed) arcs or links in G. We use again the notations and the flow feasibility conditions of section 2.1.5.

We compare the NE game with the following multi-objective model:

The variable of this model is θ with objectives $\Psi^k(\theta)$ defined by

$$\Psi^k(\theta) = \min_v C^k(v) \quad s.t. \quad v \text{ is user equilibrium w.r.t. } \beta t(v) + \theta$$

where $\beta t(v)$ is the vector of link travel time cost (compare with the NE game of section 3.2.2).

But then, since for a given θ the user equilibrium $v(\theta)$ is uniquely determined by θ, we have that

$$\Psi^k(\theta) = C^k\left(v\left(\theta\right)\right)$$

Therefore, in this model, a toll vector $\bar{\theta}$ is Pareto optimal (or non-dominated) if and only if there does not exist any other solution vector θ such that the following holds:

$$\begin{aligned} C^k\left(v\left(\theta\right)\right) &\leq C^k(v\left(\bar{\theta}\right)) \qquad \forall k \in K \quad \text{and} \\ C^j(v\left(\theta\right)) &< C^j(v\left(\bar{\theta}\right)) \qquad \text{for at least one}\, j \in K \end{aligned}$$

The set of all Pareto points (sometimes called efficient points) to a multi-objective optimization problem is called the Pareto or efficient frontier [34]: these solution points form the Pareto-optimal set P.

6.3 Solution methods

6.3.1 The game theoretic approach

As described in Chapter 3, the road pricing game is always formulated as a Stackelberg game where a leader (system controller) moves first, followed by sequential moves of other players (road users) [62, 63, 43]. When we have just one (or a weighted sum of distinct) objective, then it is assumed that only one leader stays at the upper level of the road pricing bi-level game. In practice, it always a difficult question to know when a trade-off between conflicting objectives is beneficial for multi-objective problems. Moreover, actors (stakeholders, leaders and actors are used interchangeably) have preferred objectives, and would want their preferred objectives to have more weights in the weighted sum optimization. So, a solution that favours one stakeholder may be to the detriment of another player. In this Chapter, we adopt the game theoretic model in Chapter 3 where each actor is modelled to control/optimize one externality. In that Chapter, we assume that various stakeholders can influence (or at least propose) the network tolls. In that situation, road users are influenced not only by just one leader as in a standard Stackelberg game, but by more than one decision maker. In the multi-leader-multi-follower game/problem, the leaders, turn by turn, make decisions (search for toll vectors that optimize their respective objectives of interest) at the upper level, thereby influencing the followers (users) at the lower level. A toll decision from the upper level is added to the network in the form of road tolls, consequently adding to the travel costs for these roads. The followers then react according to user or Wardrop's equilibrium - a traffic condition where no user has any incentive to switch routes. This in turn may cause the leaders to update their individual decisions (that is, changing their toll patterns) leading to lower level reactions again. Note that when an actor tolls the network in a manner that optimizes his concerned externality, the users perceive these tolls (as added travel costs) and re-route themselves to satisfy Wardrop's equilibrium. Then the next actor in turn seeing the new state of the system, and the level of tolls set by previous actors, now updates his toll (decision/strategy) to ensure that given the current situation, his present toll level is the best he can do to optimize his objective. These updates in the upper and lower level continue until a stable situation or maximum number of assigned iterations is reached. A stable (Nash equilibrium) state is reached if no stakeholder can improve his objective by unilaterally changing his proposed toll. Note, however, that given the stable state decision tolls of leaders, the lower level stable situation is given by the Wardrop's equilibrium. Therefore, the tolling game is now seen as a bi-level problem, with the stakeholders in the upper level and the travellers in the lower level. The lower level is a constraint to the upper level. In the above dynamic non-cooperative scenario, each actor continuously solves a *program with equilibrium conditions*, which is influenced by other actors' *programs with equilibrium conditions*, and these translate to an *equilibrium problem subject to equilibrium conditions*.

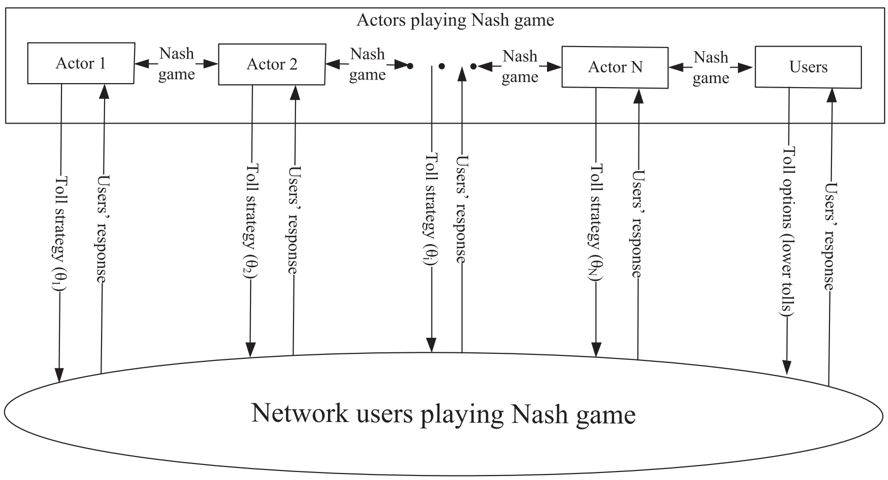


Figure 6.1: A diagram representing the dynamic game model

Note that the push by actors to optimize their objectives in every turn gives a potential to enlisting non-dominated solutions or points in every play. Our aim in this Chapter is to keep track of the attained solution during this dynamic game, construct a Pareto front from these attained solutions and compare it with the solution of the same problem solved using genetic algorithm *NSGA-II*. For the analysis of Nash equilibrium solution of the road pricing game, see Chapter 4.

Mathematical formulation of the game theoretical approach

Adopting the game model in Chapter 3, and using the Beckmann's convex formulation of Wardrop's user equilibrium (UE) [7], each actor $k \in K$ now solves the following bi-level problem:

$$\begin{gathered} \min_{\theta^k} C^k(v(\theta^k)) \\ s.t \\ FeC\_FD \\ \min_{v_a^k} \sum_{a \in A} \int_0^{v_a^k} \left(\beta t_a(u) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right) du \end{gathered} \tag{6.1}$$

Where $C^k(v(\theta^k))$ is the player k*'s* objective of interest, which depends on the network flow pattern $v(\theta^k)$, θ^k is the link toll vector of player $k \in K$, and $\sum_{j \in K \setminus k} \bar{\theta}^j$ denote cumulative toll vectors in $K \setminus k$. Note that player k cannot change this sum. Instead, given this sum, he optimizes his objective using θ^k. The first constraint *FeC_FD* ensures that the resulting flow is feasible, while the second (also called the lower level problem) ensures that the feasible flow is in user (or Wardrop's) equilibrium [7]. To simplify notations, we will mostly write v to mean $v(\theta)$.

Since the outcome of the lower level problem of Eq.(6.1) determines the input vector v^k for the objective $C^k(v^k)$ and knowing that this determinant (lower level problem) is given by the Beckmann's formulation in Eq.(6.1), player $k \in K$ thus chooses his toll θ^k in a way that optimizes his objective $C^k(v^k)$. In fact, Eq.(6.1) yields a feasible link flow vector v for every vector sum $\sum_{k \epsilon K} \bar{\theta}^k$.

For every play and for every turn, the corresponding objective values for all considered objectives are saved during the game.

6.3.2 Genetic algorithmic approach

The *NSGA-II* algorithm, developed in [16], is a multi-objective optimization algorithm that optimizes several objectives simultaneously, searching for a set of non-dominated solutions, or the Pareto optimal set. It is a genetic algorithm, so based on the principles of natural selection within evolution, it combines solutions to new solutions (*crossover*), where the solutions with higher fitness values have higher chances to survive over worse solutions. In the next generation, these enhanced solutions are recombined again, until no progress is made any more or until the maximum number of iterations H is reached. Within *NSGA-II*, the mating selection is done by binary tournament selection with replacement. All selected parents mate using uniform crossover as the crossover operator. In addition to this mating process, a random mutation operator is applied to a limited number of solutions from each generation, to promote the exploration of different regions in the solution space. In our case, mutation rate was set to 0.03, so for every design variable, there is a 0.03 chance that it is mutated. If a design variable is mutated, it is randomly set to a new feasible value. Evolutionary algorithms are often used to solve multi-objective problems, because they do not end up in a local minimum, and do not require the calculation of a gradient, and still are able to produce a diverse Pareto set. More information on genetic algorithms in a multi-objective context can be found in [15].

Within the algorithm, the fitness value is calculated in two steps. In the first step (non-dominated sorting), the solutions are ranked based on Pareto dominance. All solutions in the Pareto front receive rank 1. These solutions are then removed, and all Pareto solutions in the remaining set receive rank 2, etcetera. In the second step, the solutions are sorted within these ranks based on their crowding distance. The crowding distance calculation requires sorting of the population according to each objective value. The extreme values for each objective are assigned an infinite value, assuring that these values survive. All intermediate solutions are assigned a value equal to the absolute difference in the function values of two adjacent solutions. The crowding distance value (and thus the fitness value) is higher if a solution is more isolated, promoting a more diverse Pareto optimal set. *NSGA-II* contains elitism, to preserve good solutions in an archive φ. The archive only contains the best solutions based on the defined fitness value. This implies that in case the number of non-dominated solutions grows bigger than the archive size, solutions are selected based on crowding distance instead of dominance. For details on the algorithm, the reader is referred to [16].

The objectives optimized in system (6.1) are all system objectives, for which the Pareto set is constructed. *NSGA-II* is designed to construct a diverse set, so containing solutions with low (assuming that objectives correspond to costs) values for the first objective, but also solutions with low values for the other objectives. It aims at showing the complete spectrum of possible solutions, giving attention to all objectives (or players in the game approach). However, the travellers in the traffic system optimize their own benefits (costs in the form of tolls and travel time) in a similar way as in the game approach: they achieve user equilibrium. Therefore, the toll design problem is now seen as a bi-level problem, with the road authority in the upper level and the travellers in the lower level. The lower level is a constraint to the upper level. For every solution the genetic algorithm comes up with, a lower level user equilibrium problem is solved, resulting in network flows and costs, from which the objective functions can be calculated. This process is then repeated over and over again until no progress is made any more or until the maximum number of iterations is reached. Using *NSGA-II* as a yardstick, the results of our game model are then compared to those of the *NSGA-II*.

Mathematical formulation of genetic algorithm approach

Mathematically, the toll optimization problem for the *NSGA-II* is different from the game theoretic approach given in Eq.(6.1) in the sense that the tolls are not differentiated between the objectives. *NSGA-II* selects one generic toll θ per link to optimize the objectives simultaneously. For *NSGA-II*, the modified version of Eq.(6.1) is formulated as follows:

$$\begin{gathered}
\underset{\theta}{min}\left(C^1(v(\theta)), C^2(v(\theta)), \cdots, C^{|K|}(v(\theta))\right) \\
s.t \\
FeC_FD \\
\underset{v_a}{min} \sum_{a \in A} \int_0^{v_a} \left(\beta t_a(u) + \theta\right) du
\end{gathered} \tag{6.2}$$

FeC_FD is as given in Eq.(7.76), and $|K|$ denotes the total number of objectives (corresponding to players in the game approach). Again, the second constraint (the lower level problem) is the Beckmann's convex formulation of Wardrop's user equilibrium (UE). It ensures that any feasible solution flow v resulting from system (6.2) is in a user equilibrium: a condition where no individual road user reduces his or her travel cost by unilaterally switching routes.

Note that *NSGA-II* has been applied successfully by researchers to solve a multi-objective optimization problem in traffic engineering, e.g. [75, 60].

6.4 Numerical results

6.4.1 Link attributes and input

We will use a five-node network to compare the two models described in the preceding sections. We demonstrate the first-best pricing scheme - where tolls are allowed on all links. For the second-best scheme - where some links are not allowed to be tolled, one only needs to add the additional toll constraints on the links.

The origin-destination demand for the example network is 1000 users.

Table 6.1: Network Attributes

Link Attributes (vehicle class: private cars)

Links	Length (km)	Free Speed (km/hr)	Link capacity	Emission cost NO_x(€/gram)	Emission cost PM_{10}(€/gram)	Safety factor (injury per veh-km)
1	10	100	400	10	5	0.008
2	7	70	300	10	5	0.08
3	10.5	100	350	45	40	0.008
4	5	70	200	60	60	0.00001
5	4	70	250	45	40	0.00001
6	10	90	250	10	5	0.09
7	5	80	250	10	5	0.009
8	8.5	90	300	45	40	0.009

Emission factors (g/km/veh)

Speed (km/hr)	NO_x	PM_{10}
< 15	0.702	0.061
≤ 30	0.456	0.059
≤ 45	0.48	0.059
< 65	0.227	0.035
≥ 65	0.236	0.043

We have chosen three externalities/objective namely:

System Travel Time Cost:

$C^t(v) = \sum\limits_{a \in A} \beta v_a t_a(v_a) = \sum\limits_{a \in A} \beta v_a T_a^{ff} \left(1 + \eta \left(\frac{v_a}{\hat{C}_a}\right)^{\phi}\right);$

the so called *Bureau for Public Roads (BPR) function,* where
T_a^{ff} - free flow travel time on link a,

v_a - total flow on link a,
$\hat{C}_a$ - practical capacity of link a, and
$\eta\, and\, \phi$ - BPR scaling parameters, with $\eta = 0.15$, $\phi = 4$.
β is the value of time (VOT) with the value *0.167EUR / minute* [4], see Table 6.1 for other parameters.

Emission Cost:
$C^e(v) = \sum_{a \epsilon A} v_a \alpha_a \varkappa_a l_a$; where
$\varkappa_a$ - emission factor for link a (depending on the emission type and the vehicle speed on link a given in g/vehicle-kilometre).
l_a - length of link a. In this case study, we only consider two emission types; NO_x and PM_{10}.
See Table 6.1 for the emission costs α_a and emission factor $\varkappa_a$.

Safety Cost:
$C^s(v) = \sum_{a \epsilon A} v_a \varrho \kappa_a l_a$; where
κ_a - risk factor for link a, measured in the number of injury-crashes/vehicle-kilometre (see Table 6.1).
$E_a = l_a * v_a$ - measure of level of exposure on link a.
We set the cost of one injury ϱ to *300EUR / injury*.
Emission factors are from the CAR-model [25], emission and injury costs are chosen in a reasonable way.

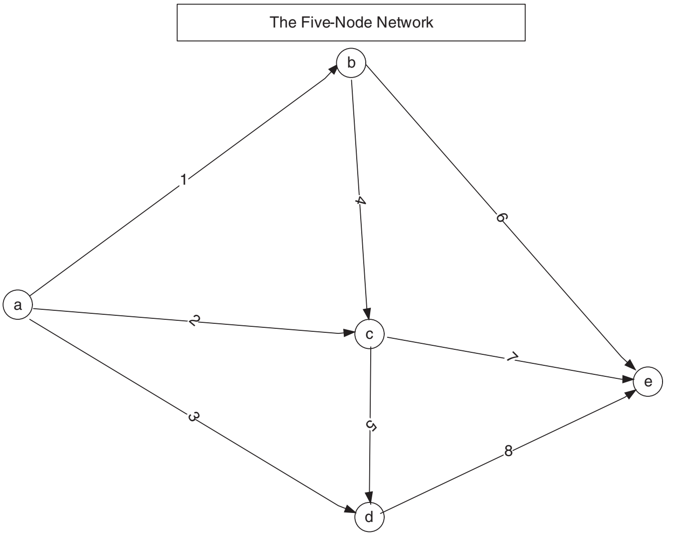


Figure 6.2: The five-node network with eight links

MATLAB is used to solve all programs. We solve the non-cooperative (Nash) game between the actors using the *NIRA-3* [31]. *NIRA-3* is a MATLAB package that uses the Nikaido-Isoda function and relaxation algorithm to find unique Nash equilibria in infinite games. An interested reader may also wish to see [30] for an evolutionary algorithm for equilibrium problem with equilibrium constraints

(EPECs). In *NIRA-3*, we set *alphamethod = 0.5, precision = [1e-3, 1e-3], and TolCon = TolFun = TolX = 1e-3*. For more on the *NIRA-3* see [31].

For the game, we place a toll bound condition of [0,5]EUR per link per player to limit the solution space. Since *NSGA-II* has discrete design variables as input, the tolls are discretised with steps of 0.1 EUR. In the game problem, each of the three players could vary the toll within the interval [0, 5], making the total toll of the three players to vary within 0 EUR and 15 EUR per link. In *NSGA-II* application, this translates to eight design variables (one for each link) with 151 different possible toll values from the set {0.0, 0.1, ... ,14.9, 15.0} for each link. Note that in *NSGA-II* application, the tolls are not differentiated between the objectives unlike in the game approach where each player has control over a specific toll range, i.e [0, 5] per link. To search for non-dominant solutions, the *NSGA-II* application uses a whole (discritised) toll range of [0, 15] per link, which corresponds to three players total toll range per link in the game approach.

Within this application of *NSGA-II*, every solution in the parent generation will combine to new solutions, so the crossover parameter is set to 1 (a chance of 1 to crossover). The initial chance for a toll value to mutate is set to 0.03. For every generation, this chance is reduced by 5%, in order to achieve convergence.

All three objectives are simultaneously optimized in *NSGA-II*, and all the three players compete in turns in the non-cooperative game. All calculations were conducted on MATLAB version 9 running on a 64-bit Windows 7 machine with 4 GB of RAM.

6.4.2 Results

Definitions

Output definitions: The set Θ is defined as all decision vectors (or solutions) that are calculated during one optimization process, so $|\Theta| = \varphi H$. Where φ is the size of the archive in one optimization process, and H is the maximum number of generations. N is the cardinality of the Pareto set. The set of N solutions $P \in \Theta$ with $P = \{v_1, v_2, \cdots, v_N\}$ is defined as the Pareto set resulting from one optimization process, which includes all non-dominated solutions with respect to all solutions in Θ, there is no $v_i \in P$ such that $v_j \in \Theta$ dominates v_i. P is the outcome of our MOP.

Hypervolume indicator: This is the space coverage of the Pareto set as implemented in [73], also known as S-metric or hypervolume. In the 2-dimensional case, it determines the area that is covered by the Pareto set with respect to a reference point (the star in Figure 6.3). The reference point represents the upper bound of all objectives: the reference point is defined such that it is dominated by all solutions in the Pareto set. Because the true maximum values of the objective functions are not known, we choose a conservative point, based on the evaluated solutions. In the 3-dimensional case, area is replaced by volume, and in the more dimensional case by hypervolume. The area or the hypervolume covered by the Pareto set P is denoted by *SSC(P)* in the figure below.

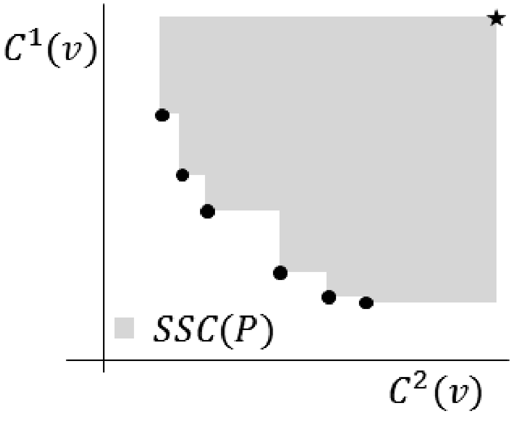


Figure 6.3: Hypervolume 2-D visualisation

In the non-cooperative game model, for every play and for every turn, the corresponding objective values for all considered objectives are saved. Similarly, the multi-objective optimization results from the *NSGA-II* are saved for every iteration. For easy visualization, we have displayed the results of the 3-dimensional optimization process for only two objectives per plot. For the *NSGA-II*, we allowed 60 solutions to be generated within one generation for hundred generations. For comparison reasons, we also allowed a maximum of 2000 play turns for each of the three players in the non-cooperative game model. On the graphs that follow, we have displayed and compared non-dominated solutions resulting from the two distinct approaches.

Figure 6.4 shows the Pareto set (or non-dominated solution) plot of the objectives; *total travel time cost* and *total safety cost* for the *NSGA-II* and game approach. See that the shapes of the two Pareto plots somewhat take the same U-shape. The plots show that *NSGA-II* generated more points in Pareto set. Furthermore, *NSGA-II* achieves better values for the safety objective. Apparently, the *Safety* player is not capable of achieving much better values while competing with other players in the game approach. This may be due to the fact that *Travel time* and *Emission* objectives are more in line with the travel cost function determining the user equilibrium (lower level problem), whereas the *Safety* objective is rather unrelated to the users' criteria. Therefore, during the game, it is easier for the *Travel time* and *Emission players* to achieve better solutions for themselves as compared to those of the *Safety player*. This indicates that the objective *safety* can only be further minimized if all three players agree to cooperate to have free access to the complete range of feasible tolls as in *NSGA-II*. That notwithstanding, the game approach does not fail in producing non-dominated solutions as we can see from the Pareto set plots. Recall that we have displayed the results of the 3-dimensional optimization process for only two objectives per plot, so some points that seem dominated are actually projections of non-dominated solutions.

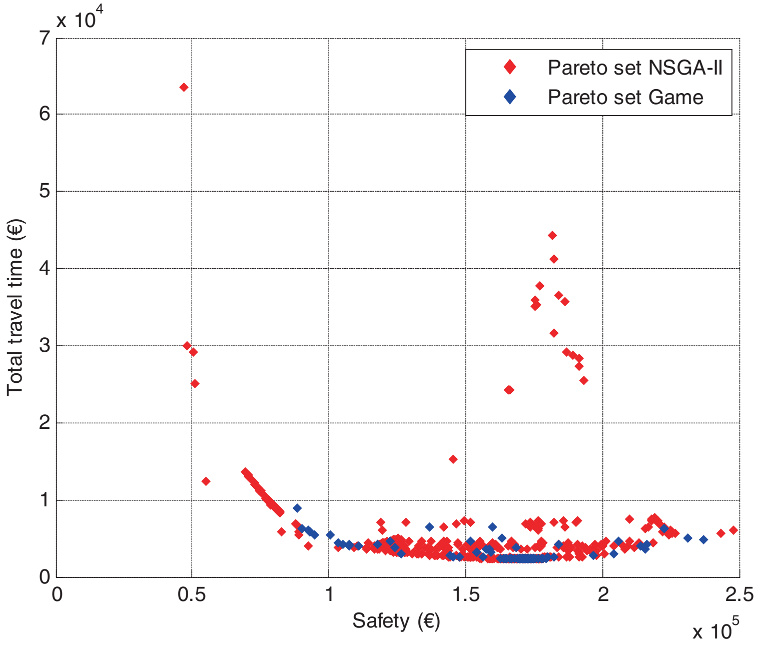


Figure 6.4: Pareto set of *Travel time cost* vs *Safety cost* from NSGA-II and Non-cooperative game

Similar as in Figure 6.4, we have displayed again a Pareto set plot of *total travel time cost* and *total emission cost* in Figure 6.5. The figure shows once more a more diverse plot by *NSGA-II*, note, however, that the differences seen in one figure are the same for all other figures, but displayed in different axis. Figure 6.6 displays the Pareto plots in the axis of *total emission cost* and *total safety cost*. What is interesting from the figures is that the game approach is almost able to discover all non-dominated plot clusters as displayed by the *NSGA-II*. We mention here that the game model is more constrained than the *NSGA-II* counterpart in the sense that *NSGA-II* has access to a whole toll spectrum [0, 15] per link to optimize the three objectives simultaneously, whereas the game approach restricts a toll range of [0, 5] per player per link. If we design the game as a cooperative game instead of the non-cooperative game, then the three players will now have access to an entire toll spectrum [0, 15] per link to optimize their three objectives at the same time just as in *NSGA-II*. However, our aim in this Chapter is to demonstrate that non-cooperative game model presents a promising way of solving multi-objective problems. In fact, the *NSGA-II* seems to be solving the cooperative game version of the game approach where all the players cooperate, use their combined toll ranges, and simultaneously do what is beneficial for all players.

Despite the "constrained" nature of the game approach, the non-cooperative game approach is capable of producing non-dominated solutions comparable to the *NSGA-II* results. This reveals that the game approach has a great potential in enlisting non-dominated solutions for multi-objective problems. Note that in

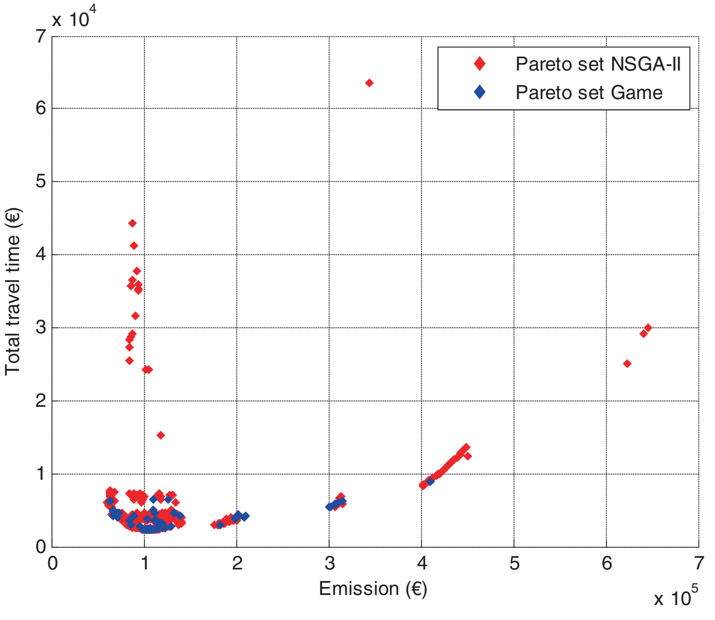


Figure 6.5: Pareto set of *Travel time cost* vs *Emission cost* from NSGA-II and Non-cooperative game

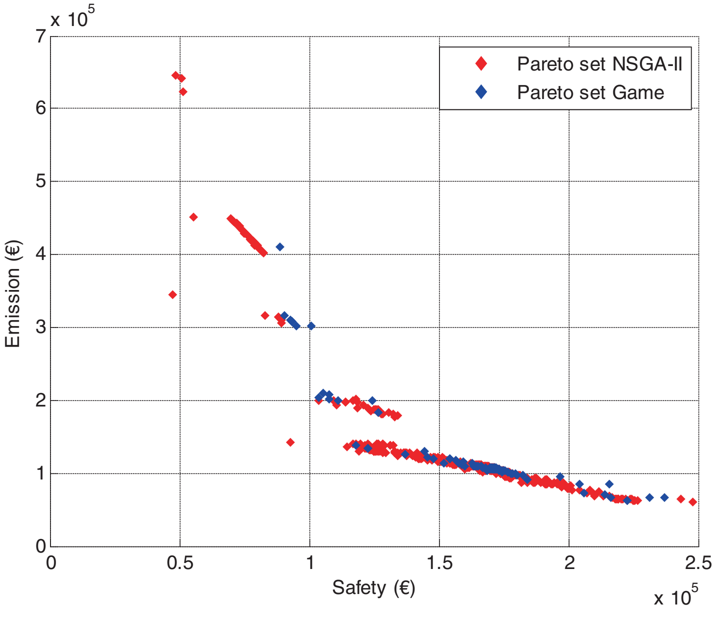


Figure 6.6: Pareto set of *Emission cost* vs *Safety cost* from NSGA-II and Non-cooperative game

general, with more solutions and iterations allowed, both the *NSGA-II* and the game approach have the potential of improving on the Pareto fronts. For the two approaches, we show below (Figure 6.7 and Figure 6.8) plots of all the generated solutions and a summary table (Table 6.2). The plots show that a range of non-dominated solutions is generated during the game. Furthermore, Figure 6.7 shows that almost all generated solutions are in the neighbourhood of the Pareto set, indicating that the non-dominated solutions are generated early in the optimization process, and further asserts the consistency of the game approach. This is also underpinned by the notion that some of the solution points replicated themselves many times during the game, due to the nature of the game where after some moves, a player will prefer to choose a set of tolls he had chosen earlier in the game. This further indicates that the game already reached convergence in less than 6000 iterations. As a result, the game approach generated a smaller number of Pareto solutions. In contrast, *NSGA-II* covers a larger solution area or hypervolume, good portions of its generated solutions are very far from the Pareto front though. However, NSGA-II in the end achieves a more diverse and richer Pareto set, as indicated by the lower (and thus better) values for the minimum objective function values for all three objectives, the higher value for hypervolume covered and the comparison plots.

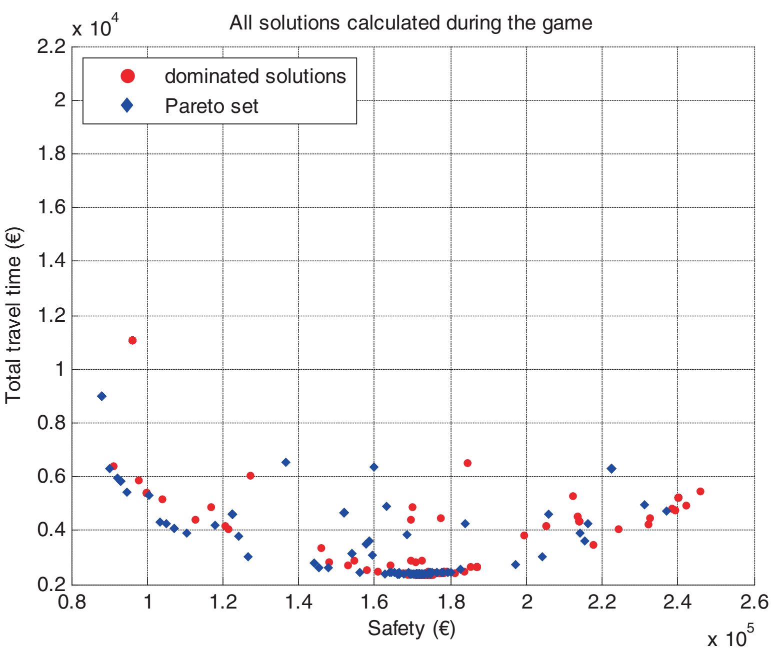


Figure 6.7: All solutions from Non-cooperative game approach

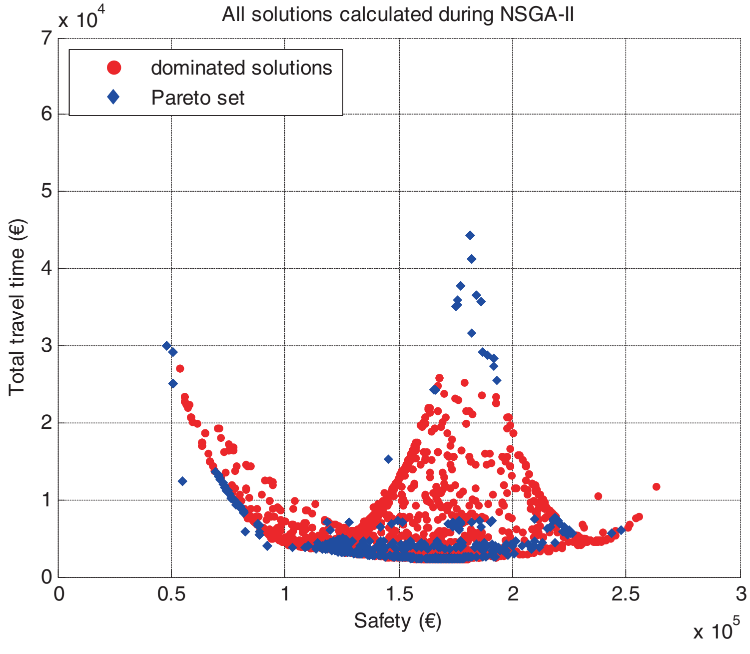


Figure 6.8: All solutions from *NSGA-II* approach

The summary table further shows that the game approach generated fewer Pareto points. Note, however, that some of these solution points (some of which are Pareto points) replicated themselves many times during the game. This is due to the nature of the game (as earlier mentioned) where after some moves, a player will prefer to choose a set of tolls he had chosen earlier in the game.

Table 6.2: Solution summary

	Game	NSGA-II
Number of solutions in the Pareto set	101	794
Minimum value for travel time	2391	2389
Minimum value for emission	63367	60930
Minimum value for safety	88240	46894
Hypervolume covered by the set	6.22E+15	7.08E+15
Time taken to run (mins)	3	11

6.5 Conclusion

In this Chapter, we compared the results of a multi-objective optimization using two distinct approaches, namely; the well-known genetic algorithm *NSGA-II* and a model from non-cooperative game theory. We applied these techniques to the problem of optimal toll design in a transportation network, with *total travel time*, *total emission cost* and *total cost of safety* as objectives. In the game theoretic approach, every objective is optimized by one of the players, while the travellers aim for user equilibrium. The results show that the game approach has the potential of discovering non-dominated solutions and can be used to solve multi-objective optimization problems. Though *NSGA-II* produces a more diverse Pareto set (seen in the plots and based on the hypervolume indicator), the game theoretic approach tends to approximate the *NSGA-II* solutions. The figures showed that similar clusters of Pareto points could be discovered by the game approach, except for the objective safety, because safety directly competes with the interests of the users. Further, plotting all solutions generated during the game showed that most dominated solutions still lie in the neighbourhood of the Pareto front, asserting the consistency of the game approach. This implies that good solutions are generated at an early stage during the game.

Although the Nash game model does not ensure that all non-dominated solutions are generated, the competition among the actors (where each actor searches for the best solution given what other actors are doing) tends to draw the solution points near to the Pareto front. On the other hand, MOPs algorithms (for example *NSGA-II*) begin to deteriorate in efficiency as the objective number increases since these algorithms depend on *Pareto dominance* to generate Pareto solution. The game mechanism we describe does not deteriorate with the number of objective, and has nothing to do with *Pareto dominance*. We thus conclude that the game theoretical approach presents a promising method for quick generation of non-dominated solutions for multi-objective problems. We acknowledge that Nash equilibrium solutions may not be Pareto efficient though.

Chapter 7

An Origin–destination based road pricing model for static and multi-period traffic assignment problems

7.1 Introduction

Over the years, researchers have focused on link-based tolling schemes [77, 67, 79]. Implementations also have focused on link-based or route-based charges. This is mainly because with link-based tolls, theoretically, one can optimally shift traffic in time. Issues ranging from fairness to financial, and from political to practical matters have long been associated with the practicality of the link-based and route-based pricing [35]. Some of these issues can easily be dealt with in an *origin–destination* (*OD*) based pricing scheme [45]. An OD-based road pricing is when users are charged based on their origin destination (OD) information. This means that all routes connecting the same OD will be charged the same amount irrespective of their lengths.

The merits discussed in section 7.1.1 form the basis of the motivation to study this novel road pricing scheme. In addition to the best of the author's knowledge, a pricing scheme that tolls the network based on origin–destination information has not been investigated before. In the case of elastic demand, the OD-tolls regulate the overall demand. When demand is fixed, the OD-based toll has no effect on the route choice of the road users since all users belonging to one OD pair are charged the same, and the fixed total demand must be realised. In this way, users travel according to Wardrop's equilibrium regardless of the tolls. In reality, what one observes is the different travel pattern for different periods of the day: the peak and off-peak periods. These periods could be modelled as a multi-period static traffic assignment, and the proposed OD-toll scheme can play an important role in shifting periodic demands. In the case of a multi-period fixed demand, the OD-tolls regulate the overall shift of demand, i.e. departure times by congestion pricing. In the first part of this Chapter, we will focus on an elastic or variable demand model where the traffic demands for a given modelling horizon depends on the so called inverse demand (or benefit) function and on the associated travel cost [77, 79]. In the second part, we will then derive a route-based pricing scheme and subsequently, an OD-based pricing scheme for a multi-period fixed demand model.

We assume that the OD-toll information will be provided to the general public so that the amount of tolls charged for a trip is known to the trip maker prior the trip.

7.1.1 Merits of the OD-based pricing scheme

An OD-based road pricing is when users are charged based on their origin destination (OD) information. This means that all routes connecting the same OD will be charged the same amount irrespective of their lengths. In fact, postal codes can serve as the origin destination nodes. For feasibility of such scheme, it is assumed that cars travel with a tracking system (for example GPS) that identifies the position of a car at any origin or destination node. The OD-based road pricing may have the following merits:

1. Almost all large cities with successful public transport (PT) operate some kind of zone-based fare system [49]. So, most merits of the PT system can be transferred to the OD-based pricing scheme.
2. It is important for public transport since it encourages intra-mode transfer, and would increase the mode share of public transport.
3. It is could be easy to implement since every car just needs to get equipped with the instrument. One would argue that with each car equipped with a tracking system, the route and the links taken by each car are known and hence the first-best link-based pricing scheme could be implemented with the same device as for the proposed OD-based pricing scheme. The answer lies in the fairness/equity feature of the OD-based pricing scheme as explained in the following merit (merit 4).
4. When temporal road disturbances (such as accidents, road constructions and repairs) occur, one does not have to pay an extra toll for using a different route since the charging is OD-based (note that this feature is difficult to achieve in the link-based toll design). In fact, when a temporal road disturbance occurs, it will be easy to efficiently redistribute traffic with OD-based tolls. Link-based scheme lacks this flexibility since one may need to construct new toll booths in order to redistribute traffic in an efficient way. Further, implementing the link-based scheme will require that you track and keep information of all links used during a trip which involves processing huge amount of data and tampering so much with travellers' privacy, where for the OD-based scheme, one only need to keep the origin and destination information just like in the PT system in The Netherlands. In the OD-based scheme, the chain of links used to complete a trip is irrelevant.
5. It could be cheaper to implement since we do not have to set toll collection booths or electronic system in the network. Remember that the first-best pricing has the possibility of setting tolls on all road segments. Owing to the impractical nature of such system, the second-best pricing scheme has gained attention, though this latter scheme does allow some road segments to be toll free, still the cost of setting and managing a toll booth is very high. In fact, for privacy and economic reasons, anonymous rechargeable tracking systems can be used for the OD-based tolling scheme. With such anonymous tracking system, we do not need that every car owner buys the said tracking system; many users can use one and the same (anonymous) tracking equipment for travel, but one user at a time making it a bit difficult to know who went to a given place at a given time.
6. Sometimes in the link-based tolling scheme, it may be inevitable for a user to use a *tolled* link in his day to day activity, for example, due to where

he lives or works. This often leads to criticism of road pricing for feelings of unfairness. With an OD-based tolling scheme, such situation does not arise.

7. No toll booth is needed in the network.
8. An OD-based road pricing scheme is capable of shifting demands (for the multi-period fixed demand traffic assignment), thus levelling the travel demand curve over time. Hence, with the OD-based pricing, we can efficiently regulate the demand into a transportation network in case when demand is elastic, and efficiently shift users from one departure time interval to another for a multi-period fixed demand model.

7.1.2 Demerits of the OD-based pricing scheme

1. Since all routes connecting the same OD are charged the same amount irrespective of their lengths, it means that with a given optimized demand $\bar{d}\,(t^i)$ during departure time interval t^i, road users will route themselves to be in a user equilibrium UE neglecting the tolls. This is because all routes connecting the same OD pair have the same toll costs on them, hence these toll costs do not influence the route choice decision of users. Therefore, the tolls have basically no effect on the in-time routing or route split of the users in the network.
2. With onymous tracking systems installed in cars, users face privacy issues as in the public transport chip card of the Netherlands, or the Octopus card of Hong Kong.
3. It could be that some users may not perceive the OD-based pricing scheme as fair owing to the fact that it is not distance based. In fact, in the proposed OD-based scheme, to induce system optimal flow, it may require that two OD pairs of equal distances are actually charged differently. This is a general characteristics of road pricing schemes, where roads/routes are charged differently irrespective of their lengths in order to achieve a desired flow pattern.

The Chapter is organised as follows: section 7.2 provides some notations used in the first part of this Chapter and the derivation of the equilibrium conditions for the OD-based scheme in a general static transportation network. In Section 7.3, we then formulate the optimization problem for route-based and OD-based second-best pricing schemes. Further in Section 7.3, explicit formulations of the OD-based tolls are derived for two and three-link networks, and for general networks. In Section 7.4, we provide some numerical examples. The second part of the Chapter starts off with Section 7.5 where we derive the route-based and OD-based pricing schemes, and the equilibrium conditions for the multi-period fixed demand model. In Section 7.6, we present a numerical example for the multi-period OD-based road pricing scheme, and Section 7.7 concludes the Chapter.

7.2 Notations and feasibility conditions

Let $G = (N, A)$ be a network, with N the set of all nodes, and A the set of (directed) arcs or links in G. The notations are defined in Table 7.1.

7.2.1 Derivation of OD-based tolls

In this section, we will derive the OD-based road pricing model for variable demand. The derivation is similar to the one given in [79].

(Road) User problem – UP

Without loss of generality, we assume that a road user only considers the costs he incurs and the benefits he enjoys making a trip. In this way, the only determinant of user's route choice behaviour is the travel costs and benefits of a trip.

Mathematically, the user problem can be formulated as a variational inequality (VI) problem [79, 19, 50, 11, 77] (see also User Problem in section 2.1.6). A given flow and demand vector $\left(\bar{f}, \bar{d}\right)$ in user equilibrium if and only if

$$\sum_{w}\left(\sum_{r}\alpha\left(\bar{\eta}_r^w\left(\bar{f}\right)\right)\left(f_r^w-\bar{f}_r^w\right)-B^w\left(\bar{d}^w\right)\left(d^w-\bar{d}^w\right)\right)\geq 0 \quad \forall f_r^w \in FeC\_ED$$

where FeC_ED stands for the *Feasibility Conditions for Elastic Demand* given below in system (7.1), and $\bar{\eta}^w\left(f\right) = \underset{r\epsilon R_w}{min}\left\{\eta_r^w\left(f\right)\right\}$ is the cost of the shortest path connecting w^{th} OD pair given the traffic flow pattern f in the network G. The parameters with the bar signs "$-$" are fixed. The user problem can then be written as the following implicit minimization problem: find $(\bar{f}, \bar{d})$ such that $\left(\bar{f}, \bar{d}\right)$ solves the following

$$\underset{f,d}{min}\sum_{w}\left(\sum_{r}\alpha\left(\bar{\eta}_r^w\left(\bar{f}\right)\right)f_r^w-B^w\left(\bar{d}^w\right)d^w\right)$$

$$\begin{array}{rcll} & s.t & & \\ v & = & \Lambda f & [\psi] \\ \Gamma f & = & d & [\lambda] \\ f & \geq & 0 & [\rho] \\ d & \geq & 0 & [\vartheta] \end{array} \qquad FeC\_ED \qquad (7.1)$$

where all variables and parameters are as given in Table 7.1 (we recall some notations).

The first constraint states that the flow on a link is equal to the sum of all path flows that pass through this link. The second equation is the flow-OD balance constraint stating that the demand is met for each OD. The third and fourth inequalities simply state that the path flows, and OD demands are non-negative. The non-negativity of link flows follows directly from the fourth constraint. $(\psi,\ \lambda,\ \rho,\ \vartheta)$ are the *KKT multipliers* associated with the constraints.

Table 7.1: Notation table

A	set of all arcs (links) in G
a	index for links
R	set of all paths
r	index for paths (routes)
W	set of all OD pairs
w	index for OD pairs
f	path flow vector
f_r^w	flow on path r belonging to the w^{th}OD pair
v	vector of link flows
Γ	OD-path incident matrix
Λ	arc-path incident matrix
V	set of feasible link flows
d	travel demand vector
d^w	demand for the w^{th}OD pair
R^w	set of all paths connecting OD pair w
$D^w(\lambda^w)$	demand function for the w^{th}OD pair
$B^w(d^w)$	inverse demand function for the w^{th}OD pair
$\eta(f)$	vector of path travel time cost functions
$\eta_r^w(f)$	travel time experienced over route $r$by users belonging to the w^{th}OD pair.
α	monetary value of time per minute(VOT)

Assumption 2:

- Throughout we assume that the route cost (or travel time) function vector $\eta(f)$ is continuous. We further assume that all constraints are differentiable and continuous and that the inverse demand functions are differentiable, separable and strictly monotonic.

We now look at the *KKT optimality conditions* of system (7.1). If we assume a separable route travel time function $\eta_r^w(f) = \eta_r^w(f_r^w)$, and let L be the Lagrangian, and $\bar{f}, \bar{d}$ be the solution to program (7.1), then there exists $(\psi, \lambda, \rho, \vartheta)$ such that the following *KKT conditions* hold:

$$
\begin{aligned}
L &= \sum_w \sum_r \alpha \left(\bar{\eta}_r^w\left(\bar{f}\right)\right) f_r^w - \sum_w B^w\left(\bar{d^w}\right) d^w + (\Lambda f - v)^T \psi + (d - \Gamma f)^T \lambda - f^T \rho \\
&\quad - d^T \vartheta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial f_r^w} L &= \alpha \bar{\eta}_r^w \left(\bar{f}\right) - \sum_a \psi_a^w \delta_{ar} - \lambda^w - \rho_r^w = 0 \;\; \forall w, r \in R^w && (7.2) \\
\frac{\partial}{\partial v_a^w} L &= \psi_a^w = 0 \qquad \forall w, a \in A && (7.3) \\
\frac{\partial}{\partial d^w} L &= \lambda^w - B^w(\bar{d}^w) - \vartheta^w = 0 \qquad \forall w && (7.4) \\
\bar{f}_r^w \rho_r^w &= 0 \qquad \forall w, r \in R^w && (7.5) \\
\bar{d}^w \vartheta^w &= 0 \qquad \forall w && (7.6) \\
\rho_r^w,\, \vartheta^w &\geq 0 \qquad \forall w, r \in R^w && (7.7)
\end{aligned}
$$

Eqs.(7.5) and (7.6) are called the *complementarity conditions.*

From Eqs.(7.2) and (7.3) we find that

$$
\alpha \bar{\eta}_r^w \;=\; \lambda^w + \rho_r^w = B^w(\bar{d}^w) + \vartheta^w + \rho_r^w \quad \text{(due to Eq. 7.4).} \tag{7.8}
$$

If the flow on route $r \in R^w$ is positive, that is $\bar{f}_r^w > 0 \Rightarrow \bar{d}^w > 0$, then the complementarity conditions in Eqs.(7.5) and (7.6) force the variables ρ_r^w and ϑ^w in Eq.(7.8) to be zero. To simplify the notation, we write $\bar{\eta}_r^w$ for $\bar{\eta}_r^w\left(\bar{f_r}^w\right)$. Recall that α is the monetary value of time (VOT). Thus, we have the following

$$
\boxed{\alpha \bar{\eta}_r^w = B^w(\bar{d}^w) \qquad \forall \bar{f}_r^w > 0,\, r \in R^w, w \in W} \tag{7.9}
$$

Interpretation: At equilibrium, the travel costs on all used routes for a given OD pair $w \in W$ are the same and equal to the benefit $B^w(\bar{d}^w)$ associated with that trip.

Due to Eq.(7.7), the following holds in general:

$$
\boxed{\alpha \bar{\eta}_r^w \geq B^w(\bar{d}^w) \qquad \forall r \in R^w,\, w \in W} \tag{7.10}
$$

Eq.(7.10) states that, at equilibrium, the travel cost on all routes for a given OD pair $w \in W$ is greater or equal to the benefit derived from making the trip. Recall from (7.9) that $B^w(\bar{d}^w)$ is the benefit on all used paths of $r \in R^w$. We thus state the following: *at equilibrium, the journey cost on all used paths/routes for a given OD pair are the same and equal to the benefit derived from making the trip, but also less than those which would be experienced by a single vehicle on any of the unused paths (Wardrop's first principle).* Therefore, we conclude that any path flow vector $\bar{f}_{R^w}^W = (\bar{f}_r^w,\, r \in R^w,\, w \in W)$ that solves system (7.1) is a user equilibrium flow.

From Eq.(7.8) we get

$$\begin{aligned}
\alpha\bar{\eta}_r^w &= B^w(\bar{d}^w) + \vartheta^w + \rho_r^w \\
\sum_r \alpha\bar{\eta}_r^w \bar{f}_r^w &= \sum_r \left(B^w(\bar{d}^w) + \vartheta^w + \rho_r^w\right) \bar{f}_r^w \\
&= \sum_r \left(B^w(\bar{d}^w) + \vartheta^w\right) \bar{f}_r^w \quad \text{(due to Eq. 7.5)} \\
&= \left(B^w(\bar{d}^w) + \vartheta^w\right) \sum_r \bar{f}_r^w \\
&= \left(B^w(\bar{d}^w) + \vartheta^w\right) \bar{d}^w \\
&= \left(B^w(\bar{d}^w)\right) \bar{d}^w \quad \text{(due to Eq. 7.6).}
\end{aligned} \tag{7.11}$$

Thus

$$\boxed{\sum_r \alpha\bar{\eta}_r^w \bar{f}_r^w = \left(B^w(\bar{d}^w)\right) \bar{d}^w \qquad \forall w \in W} \tag{7.12}$$

Eq.(7.12) is called the *network cost balance equation.*

Hence, we summarize the optimality conditions as follows:

$$\boxed{\begin{aligned}
\alpha\bar{\eta}_r^w \geq B^w(\bar{d}^w) \qquad & \forall r \in R^w, w \in W \\
\sum_r \alpha\left(\bar{\eta}_r^w\right) \bar{f}_r^w = \left[B^w(\bar{d}^w)\right] \bar{d}^w \qquad & \forall w \in W
\end{aligned}} \tag{7.13}$$

We will henceforth refer to Eq.(7.13) as *equilibrium condition (EC).* A similar result for the static traffic assignment can be found in [79, 43].

Corollary 1. *Any route flow vector ($\bar{f}_r^w$, $r \in R^w$, $w \in W$) satisfying Eq.(7.13) is a user equilibrated flow.*

Proof. The proof follows from the analysis of the *KKT optimality conditions.* ■

Decision maker's problem (system problem – SP)

The aim or objective of the system controller is to keep the social welfare as high as possible:

max [Social Welfare (or Economic Benefit (EB))]

s.t

flow feasibility conditions

The Social Welfare or EB is the difference between the **User Benefit (UB)** and **Social Cost (SC)**

$$EB = UB - SC.$$

where UB is defined as

$$UB = \sum_{w \in W} \int_0^{d^w} B^w(\varsigma) d\varsigma$$

where $B^w(d^w)$ is the inverse demand or benefit function for the OD pair $w \in W$ [77].

The **social cost SC** can be travel time cost, noise cost, emission cost, etcetera. In this specific derivation, we will take SC to be the travel time cost $\alpha\eta^T f$, where α is the value of time (VOT), η is the vector of route travel time functions, and f is the vector of route flows.

The *system problem* **SP** can then be stated mathematically as follows:

$$\begin{gathered} \underset{f,d}{min}\left(\sum_w\sum_r \alpha\left(\eta_r^w\left(f\right)\right)f_r^w - \sum_{w\in W}\int_0^{d^w} B^w(\varsigma)d\varsigma\right) \\ s.t \\ FeC\_ED \end{gathered} \tag{7.14}$$

Given that (f^*, d^*) solves SP above, then, following the same lines of arguments as in the previous section on the analysis of the *KKT optimality conditions*, we arrive at the following conditions:

$$\boxed{\begin{aligned} \alpha\left(\eta_r^{w*} + f_r^{w*}\frac{d}{df_r^w}\left(\eta_r^{w*}\right)\right) \geq B^w(d^{w*}) \qquad \forall r \in R^w, w \in W \\ \sum_r \alpha\left(\eta_r^{w*} + f_r^{w*}\frac{d}{df_r^w}\left(\eta_r^{w*}\right)\right)f_r^{w*} = \left[B^w(d^{w*})\right]d^{w*} \qquad \forall w \in W \end{aligned}} \tag{7.15}$$

7.3 Pricing schemes

7.3.1 Route based pricing scheme

A look at Eqs.(7.13) and (7.15) reveals that the only difference between the *KKT optimality conditions* is the presence of the term $\alpha f_r^{w*}\frac{d}{df_r^w}\left(\eta_p^{w*}\right)$ in the SP problem analysis. Therefore, by perturbing the route travel time η_r^w with the term $\alpha f_r^{w*}\frac{d}{df_r^w}\left(\eta_r^{w*}\right)$, users will now use the network in such a way that the resulting flow coincides with the system optimal flow pattern f^*, d^*.

This means that if every road user travelling between origin destination pair w is charged the amount $\alpha f_r^{w*}\frac{d}{df_r^w}\left(\eta_r^{w*}\right)$ for using route $r \in R^w$, then, it turns out that the 'rational' route choice decisions $\bar{f}$, $\bar{d}$ by users before embarking on a trip will coincide with the optimal route flow f^*, d^*. In fact, the term $\alpha f_r^{w*}\frac{d}{df_r^w}\left(\eta_r^{w*}\right)$ is the marginal social cost pricing (MSCP) for route $r \in R^w$. It is the extra cost for users of route $r \in R^w$ due to an additional user on this route.

For link-based pricing, [79] proved that there exist many link toll vectors that yield the system optimal flow. The same holds for the route-based toll; in fact, any route toll vector θ_r satisfying the linear conditions below will lead to the system optimal vector (f^*, d^*):

$$\boxed{\begin{aligned} \left(\alpha\eta_r^{w*} + \theta_r\right) \geq B^w(d^{w*}) \qquad \forall r \in R^w, w \in W \\ \sum_r \left(\alpha\eta_r^{w*} + \theta_r\right)f_r^{w*} = \left[B^w(d^{w*})\right]d^{w*} \qquad \forall w \in W \end{aligned}} \tag{7.16}$$

7.3.2 The OD-based pricing scheme

Issues ranging from fairness to financial issues, and from political to practical matters have long been associated with the link-based and route-based pricing [35]. Some of these issues can easily be dealt with in the proposed OD-based pricing scheme as discussed in subsection 7.1.1. In this subsection, we present a mathematical model for generating *origin–destination tolls* for every OD.

The problem is formulated as a system problem that uses an OD-based toll to change users' route choice behaviour towards a system optimal route choice pattern whilst ensuring *Wardrop's (or User) equilibrium.* The optimization program is as follows:

$$\begin{aligned} \underset{v,d,\theta}{min} \left(\sum_w \sum_r \alpha\left(\eta_r^w\left(f\right)\right) f_r^w - \sum_{w\,\epsilon\,W} \int_0^{d^w} B^w(\varsigma) d\varsigma \right) \\ s.t \\ FeC\_ED \\ \alpha\eta_r^w + \theta^w \geq B^w(d^w) \quad \forall r \in R^w, w \in W \\ \sum_r \left(\alpha\eta_r^w + \theta^w\right) f_r^w = \left[B^w(d^w)\right] d^w \quad \forall w \in W \end{aligned} \tag{7.17}$$

The objective solves the system problem, maximizing the social welfare or the economic benefit. The first constraint ensures that the generated flow pattern is a feasible network flow (see system 7.1). The second and the third constraints ensure that the generated flow pattern is in *Wardrop's (or User) equilibrium* (see Eq.(7.67)).

Notice from the *Equilibrium conditions* in system (7.17) that the tolls are OD dependent, meaning that routes belonging to the same OD-pair are charged the same cost.

It turns out that if system problem SP (system (7.14)) and user problem UP (system (7.1)) have solutions (v^*, d^*) and $(\bar{v}, \bar{d})$ respectively, and the functions $\eta(f)$ and $B(d)$ are continuous and strictly monotonic in f and d respectively, then, program (7.17) above has a unique solution $(\tilde{f}, \tilde{d})$ with the objective value

$$\tilde{Z} = \sum_w \sum_r \alpha\left(\tilde{\eta}_r^w\right) \tilde{f}_r^w - \sum_{w\,\epsilon\,W} \int_0^{\tilde{d}^w} B^w(\varsigma) d\varsigma$$

in the interval

$$\left[\sum_w \sum_r \alpha\left(\eta_r^{w*}\right) f_r^{w*} - \sum_{w\,\epsilon\,W} \int_0^{d^{w*}} B^w(\varsigma) d\varsigma \quad , \quad \sum_w \sum_r \alpha\left(\bar{\eta}_r^w\right) \bar{f}_r^w - \sum_{w\,\epsilon\,W} \int_0^{\bar{d}^w} B^w(\varsigma) d\varsigma \right]$$

We have again used $\tilde{\eta}_r^w$ to mean $\eta_r^w\left(\tilde{f}_r^w\right)$.

The argument is very easy to see since if in Eq.(7.67), there exists a toll pattern such that $\theta_r = \theta_p \; \forall r, p \in R^w, r \neq p, \; w \in W$, then the feasible flow pattern is the same as the system optimum flow pattern (SP), and the total system welfare is Z^* (i.e $\tilde{Z} = Z^*$) which is the best one can get. On the other hand, observe that the solution vector $(\bar{f}, \bar{d})$ of the user problem UP with objective value $\bar{Z}$ is a feasible solution to program (7.17) (i.e with $\theta^w = 0; \forall w \in W$), in this case $\tilde{Z} = \bar{Z}$. Search for a better solution will force some OD tolls θ^w to be non zero, so in general $\tilde{Z} \leq \bar{Z}$. Therefore, $\tilde{Z}$ is bounded below by Z^* and above by $\bar{Z}$, or

$$Z^* \leq \tilde{Z} \leq \bar{Z}$$

7.3.3 Deriving the OD toll for special cases

Optimal OD-based congestion pricing for a two-link network

In this subsection, we will derive an explicit formula for optimal OD tolls for a two-link network. Note that the proposed OD-based tolling scheme is a ‘‘second-best’’ scheme in road pricing terminology where ‘‘second-best’’ refers to the fact that there are additional constraints on the link tolls (for example, in OD-tolling where all routes belonging to the same OD are charged the same amount). Such constraints limit access to the entire solution space, and this in general, leads to sub-optimal solutions, hence the name ‘‘second-best’’. A “first-best” solution, on the contrary, allows all links to be tolled with any value. The latter is rather impractical. For this and other reasons, researchers have turned attention to a model that tolls a subset of the network links, or in general, a model that has constraints of tolls, and termed it ‘‘the second-best pricing scheme’’ [76, 66, 69, 78].

Given a two-link network with two routes (1 & 2) and one OD pairs, i.e., each link is a route, under elastic demand, we assume that the system controller would want to keep the social benefit as high as possible using OD-based tolls to regulate the demand on the network. At equilibrium, the total cost on route 1 (average cost on route 1 plus the toll on route 1) should be equal to the total cost on route 2 (average cost on route 2 plus the toll on route 2); otherwise, people would shift from one route to the other [72]. Further, at equilibrium, the total cost experienced during the entire trip equals the benefit enjoyed for the entire trip; otherwise more (in case the total cost is less than the benefit) or less (in case the total cost is more than the benefit) people will travel. Summarizing, if we assume that both links are actually used, our toll optimization problem can thus be stated as follows (see system 7.17 with $\alpha = 1$):

$$\begin{aligned} \max_{f,\theta} & \left(\int_0^{(f_1+f_2)} B(\varsigma) d\varsigma - \eta_1 f_1 - \eta_2 f_2 \right) \\ & s.t \\ & \eta_1 + \theta = B \qquad [\xi_1] \\ & \eta_2 + \theta = B \qquad [\xi_2] \end{aligned} \tag{7.18}$$

where f_1 and f_2 are flows on routes 1 and 2 respectively, $\eta_i = \eta_i(f_i)$ is the flow dependent route cost on route i, and θ is the OD-based toll, and $B = B(f_1 + f_2)$. Recall that $\xi_1$and ξ_2 are the *KKT multipliers* associated with the two constraints respectively. If we let L be the Lagrangian, then there exist (ξ_1, ξ_2) such that the following *KKT conditions* hold for system (7.18):

$$\begin{aligned} L &= \int_0^{(f_1+f_2)} B(\varsigma)d\varsigma - \eta_1 f_1 - \eta_2 f_2 \\ &+ (B - \eta_1 - \theta)\,\xi_1 + (B - \eta_2 - \theta)\,\xi_2 \end{aligned}$$

$$\frac{\partial}{\partial f_1}L = B - \eta_1^{'} f_1 - \eta_1 - \eta_1^{'}\xi_1 + B^{'}\xi_1 + B^{'}\xi_2 = 0 \tag{7.19}$$

$$\frac{\partial}{\partial f_2}L = B - \eta_2^{'} f_2 - \eta_2 - \eta_2^{'}\xi_2 + B^{'}\xi_1 + B^{'}\xi_2 = 0 \tag{7.20}$$

$$\frac{\partial}{\partial \theta}L = -\xi_1 - \xi_2 = 0 \tag{7.21}$$

$$\frac{\partial}{\partial \xi_i}L = B - \eta_i - \theta = 0 \qquad \forall i \tag{7.22}$$

where $\eta_i^{'} = \frac{d}{df_i}\eta_i(f_i)$, and $B^{'} = \frac{\partial}{\partial f_i}B\left(\sum_i f_i\right)$.

Eq.(7.20) minus Eq.(7.19), using the fact that in equilibrium $\eta_1 = \eta_2$, together with Eq.(7.21) yield

$$\eta_1^{'} f_1 - \eta_2^{'} f_2 + \eta_1^{'}\xi_1 - \eta_2^{'}\xi_2 = 0 \tag{7.23}$$

$$\eta_1^{'} f_1 - \eta_2^{'} f_2 + \eta_1^{'}\xi_1 + \eta_2^{'}\xi_1 = 0 \tag{7.24}$$

$$\xi_1 = \frac{\eta_2^{'} f_2 - \eta_1^{'} f_1}{\eta_1^{'} + \eta_2^{'}} \tag{7.25}$$

from Eq.(7.19), (7.21) , (7.22) and (7.25)

$$B = \eta_1^{'} f_1 + \eta_1 + \eta_1^{'}\xi_1$$

$$\begin{aligned} \theta &= B - \eta_1 \\ &= \eta_1^{'} f_1 + \eta_1^{'}\xi_1 \\ &= \eta_1^{'} f_1 + \eta_1^{'}\frac{\eta_2^{'} f_2 - \eta_1^{'} f_1}{\eta_1^{'} + \eta_2^{'}} \end{aligned}$$

$$\theta = \frac{\eta_2^{'}\left(\eta_1^{'} f_1\right) + \eta_1^{'}\left(\eta_2^{'} f_2\right)}{\eta_1^{'} + \eta_2^{'}} \tag{7.26}$$

Eq.(7.26) reveals that the OD toll on each of the routes is the weighted mean of the marginal external costs, where the weight of route i is the derivative of the usage cost of route j.

Note that this demonstration generalises to two-route networks with one OD pair, where a route may contain more than one link, and in that case, $\eta_i^{'} = \frac{\partial}{\partial f_i}\eta_i(f)$, where f_i is the flow on link i.

Second-best OD-based congestion pricing for a three-link network

Given a three-link network with three routes and one OD pairs, i.e., each link is a route, we again analyse the following optimization problem, the Lagrangian and the *KKT optimality conditions*:

$$\max_{f,\theta}\left(\int_0^{(f_1+f_2+f_3)} B(\varsigma)d\varsigma - \eta_1 f_1 - \eta_2 f_2 - \eta_3 f_3\right)$$

$$s.t \tag{7.27}$$

$$\begin{aligned} \eta_1+\theta &= B \qquad [\xi_1] \\ \eta_2+\theta &= B \qquad [\xi_2] \\ \eta_3+\theta &= B \qquad [\xi_3] \end{aligned}$$

where $B = B\left(f_1+f_2+f_3\right)$.

$$\begin{aligned} L \;=\; & \int_0^{(f_1+f_2+f_3)} B(\varsigma)d\varsigma - \eta_1 f_1 - \eta_2 f_2 - \eta_3 f_3 \\ & + \left(B-\eta_1-\theta\right)\xi_1 + \left(B-\eta_2-\theta\right)\xi_2 + \left(B-\eta_3-\theta\right)\xi_3 \end{aligned}$$

$$\frac{\partial}{\partial f_1}L = B - \eta_1^{'} f_1 - \eta_1 - \eta_1^{'}\xi_1 + B^{'}\xi_1 + B^{'}\xi_2 + B^{'}\xi_3 = 0 \tag{7.28}$$
$$\frac{\partial}{\partial f_2}L = B - \eta_2^{'} f_2 - \eta_2 - \eta_2^{'}\xi_2 + B^{'}\xi_1 + B^{'}\xi_2 + B^{'}\xi_3 = 0 \tag{7.29}$$
$$\frac{\partial}{\partial f_3}L = B - \eta_3^{'} f_3 - \eta_3 - \eta_3^{'}\xi_3 + B^{'}\xi_1 + B^{'}\xi_2 + B^{'}\xi_3 = 0 \tag{7.30}$$
$$\frac{\partial}{\partial \theta}L = -\xi_1 - \xi_2 - \xi_3 = 0 \tag{7.31}$$
$$\frac{\partial}{\partial \xi_i}L = B - \eta_i - \theta = 0 \qquad \forall i \tag{7.32}$$

Eq.(7.29) minus Eq.(7.28), (7.30) minus (7.29), (7.30) minus (7.28), together with the fact that in equilibrium $\eta_1 = \eta_2 = \eta_3$ yield

$$\eta_1^{'} f_1 - \eta_2^{'} f_2 + \eta_1^{'}\xi_1 - \eta_2^{'}\xi_2 = 0 \tag{7.33}$$
$$\eta_2^{'} f_2 - \eta_3^{'} f_3 + \eta_2^{'}\xi_2 - \eta_3^{'}\xi_3 = 0 \tag{7.34}$$
$$\eta_1^{'} f_1 - \eta_3^{'} f_3 + \eta_1^{'}\xi_1 - \eta_3^{'}\xi_3 = 0 \tag{7.35}$$

from Eq.(7.33)

$$\begin{aligned}
\eta_1^{'}\xi_1 &= \eta_2^{'}f_2 - \eta_1^{'}f_1 + \eta_2^{'}\xi_2 \\
\xi_1 &= \frac{\eta_2^{'}f_2 - \eta_1^{'}f_1 + \eta_2^{'}\xi_2}{\eta_1^{'}}.
\end{aligned} \tag{7.36}$$

Then Eqs (7.34) and (7.31) imply

$$\begin{aligned}
\eta_2^{'}f_2 - \eta_3^{'}f_3 + \eta_2^{'}\xi_2 - \eta_3^{'}\xi_3 &= 0 \\
\eta_2^{'}f_2 - \eta_3^{'}f_3 + \eta_2^{'}\xi_2 + \eta_3^{'}\left(\xi_1 + \xi_2\right) &= 0 \\
\eta_2^{'}f_2 - \eta_3^{'}f_3 + \eta_3^{'}\xi_1 + \left(\eta_2^{'} + \eta_3^{'}\right)\xi_2 &= 0 \\
\xi_1 &= \frac{\eta_3^{'}f_3 - \eta_2^{'}f_2 - \left(\eta_2^{'} + \eta_3^{'}\right)\xi_2}{\eta_3^{'}}
\end{aligned} \tag{7.37}$$

Comparing Eqs (7.36) and (7.37) we find that

$$\begin{aligned}
\frac{\eta_3^{'}f_3 - \eta_2^{'}f_2 - \left(\eta_2^{'} + \eta_3^{'}\right)\xi_2}{\eta_3^{'}} &= \frac{\eta_2^{'}f_2 - \eta_1^{'}f_1 + \eta_2^{'}\xi_2}{\eta_1^{'}} \\
\xi_2 &= \frac{\eta_1^{'}\eta_3^{'}f_3 + \eta_3^{'}\eta_1^{'}f_1 - \eta_1^{'}\eta_2^{'}f_2 - \eta_3^{'}\eta_2^{'}f_2}{\left(\eta_1^{'}\eta_2^{'} + \eta_1^{'}\eta_3^{'} + \eta_2^{'}\eta_3^{'}\right)}
\end{aligned}$$

from Eq.(7.29)

$$\begin{aligned}
B - \eta_2^{'}f_2 - \eta_2 - \eta_2^{'}\xi_2 + B^{'}\xi_1 + B^{'}\xi_2 + B^{'}\xi_3 &= 0 \\
B - \eta_2^{'}f_2 - \eta_2 - \eta_2^{'}\xi_2 + B^{'}\left(\xi_1 + \xi_2 + \xi_3\right) &= 0 \\
B - \eta_2^{'}f_2 - \eta_2 - \eta_2^{'}\xi_2 &= 0 \\
B &= \eta_2^{'}f_2 + \eta_2 + \eta_2^{'}\xi_2.
\end{aligned}$$

Hence from (7.32)

$$\begin{aligned}
\theta &= B - \eta_2 \\
&= \eta_2^{'}f_2 + \eta_2^{'}\xi_2 \\
&= \eta_2^{'}f_2 + \eta_2^{'}our\frac{\eta_1^{'}\eta_3^{'}f_3 + \eta_3^{'}\eta_1^{'}f_1 - \eta_1^{'}\eta_2^{'}f_2 - \eta_3^{'}\eta_2^{'}f_2}{\left(\eta_1^{'}\eta_2^{'} + \eta_1^{'}\eta_3^{'} + \eta_2^{'}\eta_3^{'}\right)} \\
\theta &= \frac{\left(\eta_2^{'}\eta_3^{'}\right)\eta_1^{'}f_1 + \left(\eta_1^{'}\eta_3^{'}\right)\eta_2^{'}f_2 + \left(\eta_1^{'}\eta_2^{'}\right)\eta_3^{'}f_3}{\left(\eta_1^{'}\eta_2^{'} + \eta_1^{'}\eta_3^{'} + \eta_2^{'}\eta_3^{'}\right)}
\end{aligned} \tag{7.38}$$

Again, the OD toll on each of the routes is the weighted average of the marginal external costs, where the weight of route i is the product of the derivative of the usage cost of the other two routes. Note that this demonstration also generalises to three-route networks with one OD pair, where a route may contain more than one link, and in that case, $\eta_i^{'} = \frac{\partial}{\partial f_i}\eta_i\left(f\right)$, where f_i is the flow on link i.

Second-best OD-based congestion pricing for a general network

Using system (7.17), the optimization problem for the second-best OD-based pricing scheme for a general network is stated as follows:

$$\max_{f,d,\theta}\left(\sum_{w\,\epsilon\,W}\int_0^{d^w} B^w(\varsigma)d\varsigma-\sum_w\sum_r\alpha\left(\eta_r^w\left(f\right)\right)f_r^w\right)$$

$$s.t \tag{7.39}$$

$$\begin{aligned}
\alpha\eta_r^w+\theta^w &\geq B^w(d^w)\quad \forall r\in R^w,w\in W && [\xi_r^w]\\
\sum_r\left(\alpha\eta_r^w+\theta^w\right)f_r^w &= \left[B^w(d^w)\right]d^w\quad \forall w\in W && [\zeta^w]\\
f_r^w &\geq 0 && [\rho_r^w]
\end{aligned}$$

where $d^w=\sum\limits_{r\,\epsilon\,R^w} f_r^w$ is the total demand for the w^{th} OD pair, and θ^w is the OD toll for the w^{th} OD pair. α is the value of time (VOT).

At equilibrium, we actually require that for a given OD pair w, the marginal benefit equals the average cost plus tolls incurred on any single used path belonging to w. Therefore, the first (inequality) constraint becomes equality, and the second and third constraints follow immediately (and thus can be deleted). The Lagrangian is thus given by

$$L=\sum_{w\,\epsilon\,W}\int_0^{d^w}B^w(\varsigma)d\varsigma-\alpha\sum_w\sum_r\left(\eta_r^w\right)f_r^w+\sum_w\sum_r\left(B^w(d^w)-\alpha\eta_r^w-\theta^w\right)\xi_r^w$$

$$\begin{aligned}
\frac{\partial}{\partial f_r^w}L &= B^w-\alpha\left(\eta_r^w\right)'f_r^w-\alpha\eta_r^w+\sum_{s\,\epsilon\,R^w\backslash r}\left(B^w\right)'\xi_s^w+ && (7.40)\\
&\qquad\left(\left(B^w\right)'-\left(\alpha\eta_r^w\right)'\right)\xi_r^w=0\ \ \forall r\in R^w,w\in W\\
\frac{\partial}{\partial\theta^w}L &= -\sum_{r\,\epsilon\,R^w}\xi_r^w=0\qquad\forall w\in W && (7.41)\\
\frac{\partial}{\partial\xi_r^w}L &= B^w-\alpha\eta_r^w-\theta^w=0\qquad\forall r\in R^w,w\in W && (7.42)
\end{aligned}$$

where $\left(\eta_r^w\right)'=\frac{\partial}{\partial f_r^w}\eta_r^w\left(f\right)$, and $\left(B^w\right)'=\frac{\partial}{\partial f_r^w}B^w\left(d^w\right)$ with $d^w=\sum\limits_{r\in R^w}f_r^w$.

From Eqs.(7.40) and (7.41)

$$B^w-\alpha\left(\eta_r^w\right)'f_r^w-\alpha\eta_r^w+\sum_{s\,\epsilon\,R^w\backslash r}\left(B^w\right)'\xi_s^w+\left(B^w\right)'\xi_r^w-\alpha\left(\eta_r^w\right)'\xi_r^w=0\qquad\forall r\in R^w,w\in W$$

$$
\begin{aligned}
B^w - \alpha \left(\eta_r^w\right)' f_r^w - \alpha \eta_r^w + \left(\left(B^w\right)' \sum_{s \,\epsilon\, R^w \setminus r} \xi_s^w \right) - \\
\left(\left(B^w\right)' \sum_{s \,\epsilon\, R^w \setminus r} \xi_s^w \right) - \alpha \left(\eta_r^w\right)' \xi_r^w &= 0 \qquad \forall r \in R^w, w \in W \qquad (7.43) \\
B^w - \alpha \left(\eta_r^w\right)' f_r^w - \alpha \eta_r^w - \alpha \left(\eta_r^w\right)' \xi_r^w &= 0 \qquad \forall r \in R^w, w \in W \\
B^w = \alpha \left(\eta_r^w\right)' f_r^w + \alpha \eta_r^w + \alpha \left(\eta_r^w\right)' \xi_r^w & \qquad\qquad r \in R^w \qquad (7.44)
\end{aligned}
$$

Using Eq.(7.44), and that at equilibrium $\eta_r^w = \eta_s^w$ for $r, s \in R^w$

$$
\begin{aligned}
\frac{\partial}{\partial f_s^w} L - \frac{\partial}{\partial f_r^w} L = \left(\eta_r^w\right)' f_r^w - \left(\eta_s^w\right)' f_s^w + \left(\eta_r^w\right)' \xi_r^w - \left(\eta_s^w\right)' \xi_s^w &= 0 \; for \; any \; r, s \in R^w, w \in W \\
\left(\eta_r^w\right)' f_r^w - \left(\eta_s^w\right)' f_s^w + \left(\eta_r^w\right)' \xi_r^w + \left(\eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus s} \xi_j^w &= 0 \\
\left(\eta_r^w\right)' f_r^w - \left(\eta_s^w\right)' f_s^w + \left(\eta_r^w\right)' \xi_r^w + \left(\eta_s^w\right)' \xi_r^w + \left(\eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus r,s} \xi_j^w &= 0
\end{aligned}
$$

$$
\xi_r^w = \frac{\left(\eta_s^w\right)' f_s^w - \left(\eta_r^w\right)' f_r^w - \left(\eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus r,s} \xi_j^w}{\left(\eta_r^w\right)' + \left(\eta_s^w\right)'} \quad \forall r \in R^w, w \in W \,; s \in R^w \qquad (7.45)
$$

$$
\begin{aligned}
\theta^w &= B^w - \alpha \eta_r^w \qquad \forall w \in W \,; r \in R^w \\
&= \alpha \left(\eta_r^w\right)' f_r^w + \alpha \left(\eta_r^w\right)' \xi_r^w \qquad \forall w \in W ; \; r \in R^w \quad \text{(from Eqn (43))}
\end{aligned}
$$

$$
\theta^w = \alpha \left(\eta_r^w\right)' f_r^w + \alpha \left(\eta_r^w\right)' \frac{\left(\eta_s^w\right)' f_s^w - \left(\eta_r^w\right)' f_r^w - \left(\eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus r,s} \xi_j^w}{\left(\eta_r^w\right)' + \left(\eta_s^w\right)'} \qquad \forall w \in W ; \; r, s \in R^w
$$

$$
\theta^w = \alpha \frac{\left(\eta_s^w\right)' \left(\left(\eta_r^w\right)' f_r^w \right) + \left(\eta_r^w\right)' \left(\left(\eta_s^w\right)' f_s^w \right) - \left(\eta_r^w \cdot \eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus r,s} \xi_j^w}{\left(\eta_r^w\right)' + \left(\eta_s^w\right)'} \qquad \forall w \in W ; \; r, s \in R^w \quad (7.46)
$$

where ξ_j^w is given by Eq.(7.45). To understand the full structure of the generalised OD-toll in Eq.(7.46), one needs to explicitly determine all ξ_j^w for all j. It is worth knowing that for four-route network with one OD-pair, Eq.(7.46) reduces to a result similar to the ones derived in the two-route and three-route networks: The weighted average of the marginal external costs, where the weight of route i is the product of the derivative of the usage cost of other routes. Further analysis of Eq.(7.46) suggest the same trend as given below:

$$
\xi_s^w = \frac{\left(\eta_p^w\right)' \left(\left(\eta_p^w\right)' f_p^w \right) - \left(\eta_r^w\right)' \left(\left(\eta_s^w\right)' f_s^w \right) - \left(\eta_p^w\right)' \left(\left(\eta_s^w\right)' f_s^w \right) + \left(\eta_p^w\right)' \left(\left(\eta_r^w\right)' f_r^w \right) - \left(\eta_p^w \cdot \eta_r^w\right)' \sum_{j \,\epsilon\, R^w \setminus p,r,s} \xi_j^w}{\left(\eta_p^w \cdot \eta_r^w\right)' + \left(\eta_p^w \cdot \eta_s^w\right)' + \left(\eta_r^w \cdot \eta_s^w\right)'}
$$

$$
\forall w \in W ; \; p, r, s \in R^w \qquad (7.47)
$$

$$
\theta^w = \alpha \frac{\left(\eta_p^w \cdot \eta_s^w\right)' \left(\left(\eta_r^w\right)' f_r^w \right) + \left(\eta_p^w \cdot \eta_r^w\right)' \left(\left(\eta_s^w\right)' f_s^w \right) + \left(\eta_r^w \cdot \eta_s^w\right)' \left(\left(\eta_p^w\right)' f_p^w \right) - \left(\eta_p^w \cdot \eta_r^w \cdot \eta_s^w\right)' \sum_{j \,\epsilon\, R^w \setminus p,r,s} \xi_j^w}{\left(\eta_p^w \cdot \eta_s^w\right)' + \left(\eta_p^w \cdot \eta_r^w\right)' + \left(\eta_r^w \cdot \eta_s^w\right)'}
$$

$$
\forall w \in W ; \; p, r, s \in R^w \qquad (7.48)
$$

As a future research, we would like to see if there is an simpler way of explicitly determining the multiplier ξ_j^w for all j.

7.4 Numerical examples

In this section, we demonstrate our model so far using a simple two-route network (see Figs. 7.1 and 7.2). In our model and derivations, we have focused on travel time cost. Recall that the choice of travel time cost $\eta^T f$ is an arbitrary choice. A decision maker may as well choose to optimize emission, noise, safety etcetera. In this specific example, we optimize the entire travel cost using the OD-based toll. We assume undifferentiated users (extension to multiple user classes is straightforward).

7.4.1 Network 1

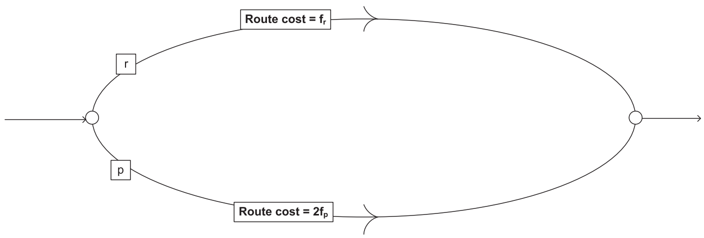


Figure 7.1: Two-route network 1

We define the following inverse demand (benefit) function for the single-OD network

$$B^w(d^w) = 20 - \frac{d^w}{2}$$

Results

Table 7.2: Result table for network 1

	f_r	f_p	d^w	$\eta(f_r)$	$\eta(f_p)$	θ_r	θ_p	$\eta(f_r)+\theta_r$	$\eta(f_p)+\theta_p$	$B^w(d^w)$	Social Welfare
User Equilibrium	11.43	5.71	17.14	11.43	11.43	0.00	0.00	11.43	11.43	11.43	73.47
System Optimum	7.27	3.64	10.91	7.27	7.27	7.27	7.27	14.55	14.55	14.55	109.09
OD-based toll	7.27	3.64	10.91	7.27	7.27	7.27	7.27	14.55	14.55	14.55	109.09

Observe also that we could achieve the system optimal flow pattern with an OD-based toll in this specific example. The tolls ensure that the system optimal flow is in *Wardrop's equilibrium*. In fact, the OD tolls coincide with the first-best tolls, leading to the optimal system welfare. The UE flow pattern over-used the network, leading to low benefit values for the users, as reflected in the low societal benefit seen in the *social welfare* column of Table 7.2. Throughout, we assume that the generated toll revenue is invested back into the transportation network so as not to increase the travel cost (see Table 7.1).

In general, the OD-based tolls coincide with those of the system optimum (SO) if the following holds in particular:

$$\begin{aligned}
\theta^w(OD) &= \theta_r^w(SO) = \theta_p^w(SO) \\
where & \\
\theta^w(OD) &= \frac{\eta_2^{'}\eta_1^{'}f_1 + \eta_1^{'}\eta_2^{'}f_2}{\eta_1^{'} + \eta_2^{'}} \\
\theta_r^w(SO) &= \eta_1^{'}f_1 \\
\theta_p^w(SO) &= \eta_2^{'}f_2
\end{aligned}$$

$\theta^w(OD)$ is the OD-based tolls for the origin–destination w given in Eq.(7.26), $\theta_r^w(SO)$ and $\theta_p^w(SO)$ are the marginal cost tolls on routes r and p respectively. Note that tolls could also be defined in a different way. Indeed, any toll vectors that satisfy only the first line of conditions given above, will induce the system optimum.

7.4.2 Network 2

Now, we alter the travel cost on route r and solve system (7.17) again.

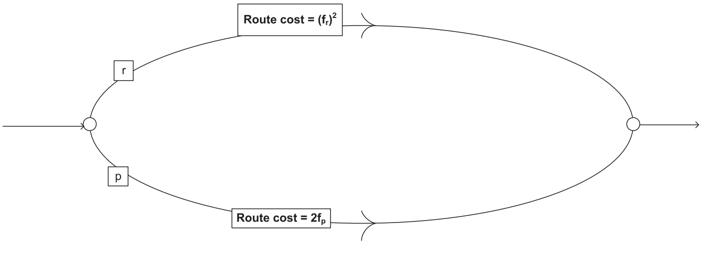


Figure 7.2: Two-route network 2

We use the same inverse demand (benefit) function for the single-OD network

$$B^w(d^w) = 20 - \frac{d^w}{2}$$

Results

Table 7.3: Result table for network 2

	f_r	f_p	d^w	$\eta(f_r)$	$\eta(f_p)$	θ_r	θ_p	$\eta(f_r)+\theta_r$	$\eta(f_p)+\theta_p$	$B^w(d^w)$	Social Welfare
User Equilibrium	3.80	7.24	11.04	14.48	14.48	0.00	0.00	14.48	14.48	14.48	30.49
System Optimum	2.36	4.18	6.54	5.58	8.36	11.15	8.36	16.73	16.73	16.73	72.02
OD-based toll	2.72	3.70	6.42	7.40	7.40	9.39	9.39	16.79	16.79	16.79	70.59

In this second network example, though the *social welfare* due to the OD pricing scheme slightly fell short (by less than 2%) of the *social welfare* of the system optimum, it (the OD scheme) improved the no-toll user equilibrium scenario by more than 130%. Notice again that the users overused the network by almost 100% in an uncontrolled network scenario (UE), leading to the low *social welfare*. With these simple examples, we demonstrate that the proposed OD-based pricing scheme has the potential of greatly improving a no-toll or uncontrolled network scenario (see 7.3).

7.5 An OD-based pricing for a multi-period static traffic assignment

7.5.1 Introduction

Having dealt with the OD-based pricing scheme for variable demand, we now turn our attention to a fixed demand model where demands are read from OD matrices built by observing traffic over time. With the input from the matrix, usually, fixed demand pricing models fix the demand for each time interval as read from the OD matrix, and search for optimal toll (and hence flow) pattern for this time slot. The proposed OD-based tolling scheme will have no effect for such model since the (OD-based) scheme does not optimize route split. To further explain, modelling (one-period) fixed demand requires that all input demand be realised for all ODs, and this requirement strips the OD-based tolling scheme of its shift demand effect. All routes belonging to the same OD pair are charged the same, so users belonging to the same OD pair will disregard the corresponding OD tolls while optimizing their route choice.

In the model we propose, we look at a given modelling time horizon T divided into multiple periods. We take input demand from the time-dependent OD matrix for the discrete and connected time periods $t^i s$, we then allow that these input demands are not fixed for the time intervals/periods, but 'elastic' within the entire modelling horizon. To further illustrate, suppose that T is divided into three periods, say t^1, t^2 and t^3, and suppose that the input or the counted demand $\hat{d}$ is as follows: $\hat{d}\,(t^1) = 200$, $\hat{d}\,(t^2) = 1000$ and $\hat{d}\,(t^3) = 300$, then using OD-based tolls, we can shift demands within T in a more efficient way. In the optimized scenario with the OD-based tolls, it may be that the optimized demand $\bar{d}$ is now given by $\bar{d}\,(t^1) = 450$, $\bar{d}\,(t^2) = 650$ and $\bar{d}\,(t^3) = 400$. We allow for this flexibility in demand because counted traffic (i.e. user behaviour or user equilibrium) may be far from the system optimal traffic flow pattern as we will see later [12]. The flexibility on demand over the modelling periods allows efficient distribution of demand across the modelling periods or horizon. On the other hand, the total amount of traffic over the entire modelling horizon T (as read from the input matrix) must be realized [28].

In steps, the OD-based pricing for a multi-period static traffic assignment (MSTA) involves: (1) Reading demand for all periods over the entire T from a time-dependent OD-matrix, (2) Allow the total demand over T be fixed/realized, and (3) Optimize the demand shifts within T using the OD-based tolls whilst ensuring user equilibrium within and across the $t^i s$. In addition, the proposed model includes the fact that observed departure travel pattern is not necessarily preferred departure pattern for the road users [61, 9]. To this end, we borrow the schedule delay idea of [61, 9, 2] to account for the cost involved in shifting departure time of a user from a given time slot to another.

7.5.2 Model formulation

Let $G = \{N, A\}$ denote a transportation network consisting of a set of nodes N and a set of links A. Let P be the set of all routes in G and $p \in P$ the index for

Table 7.4: Notations

$x_a\left(t^i\right)$	number of vehicles traversing link $a$during time t^i
$x^w_{ap}\left(t^i\right)$	number of vehicles traversing link $a$on route p between OD pair $w$during departure time interval t^i
$u_a\left(t^i\right)$	inflow rate of link $a$during departure time interval t^i
$u^w_{ap}\left(t^i\right)$	inflow rate of link $a$on route $p$between OD pair $w$during t^i
$v_a\left(t^i\right)$	exit flow rate of link $a$during departure time interval t^i
$v^w_{ap}\left(t^i\right)$	exit rate of link $a$on route $p$between w^{th}OD pair during t^i
$E^w_p\left(t^i\right)$	cummulative number of vehicles arriving destination s from origin $r$on route $p$by time t^i
$e^w_p\left(t^i\right)$	arrival flow rate at the destination for the w^{th}OD pair on route $p$during t^i
$f^w\left(t^i\right)$	depature flow rate for the w^{th}OD pair during time t^i
$f^w_p\left(t^i\right)$	depature flow rate into route $p$for the w^{th}OD pair during t^i
F	set of all feasible path flows
P_w	set of all paths belonging to the w^{th}OD pair
N	set of all nodes in the network
$A(n)$	set of links whose tail node is n
$B(n)$	set of links whose head node is n
$\tau_a\left(t^i\right)$	travel time over link $a$for flows entering link $a$during t^i
$\eta^w_p\left(f\left(t^i\right)\right)$	travel time experienced over route $p$by users belonging to the w^{th}OD pair departing during t^i
$\hat{d^w}\left(t^i\right)$	demand for the w^{th}(r-s) OD pair during time t^ias observed from the input OD matrix. It is the number of travellers departing origin $r$during t^itowards destination s
$\bar{d^w}\left(t^i\right)$	optimized demand for the w^{th}(r-s) OD pair at time t^i.It is the optimize number of travellers departing origin r during t^itowards destination s
$c^w_{t^jt^i}$	cost involved in shifting departure time of a user from $t^j$to $t^i$for w^{th}OD pair
$y^w_{t^jt^i}$	number of users who were departing during t^jbut are rescheduled to depart during $t^i$for the w^{th}OD pair in the optimized model
$z^w_{t^jt^i}$	number of users who prefer departing during t^jbut are are actually departing during $t^i$for the w^{th}OD pair

routes in G. One or more routes $p \in P$ may exist between origin (r)-destination (s) pair $rs = w \in W$. We use W to denote the set of all origin–destination (OD) pairs. Every route p is comprised of one or more links $a \in A$. We use a discrete time formulation in which the whole studied time period T is divided into a certain number of small time intervals, denoted by t^i [28]. These discrete time intervals t^i with $i = 1, 2, \ldots, T$ are such that they correspond to the departure times. For example, if the study time period T is the from 6:00h to 12:00h, then the departure time intervals t^i can be $t^1 = 6:00h - 6:15h$, $t^2 = 6:15h - 6:30h$, $t^3 = 6:30h - 6:45h$, and so on. Note that this length of the interval is arbitrarily chosen. We will consider one user class model. Heterogeneous users' model is a straightforward extension. Throughout, we omit the constant value of time (VOT) α. First, we derive models for the route-based pricing, and then the OD-based pricing scheme. The notations and the feasibility conditions given are derived from [11] (see Table 7.4).

Flow conservation constraints

$$f_p^w\left(t^i\right) = \sum_{a \epsilon A(r)} u_{ap}^w\left(t^i\right) \qquad \forall i, w, p \in P_w, \quad [\alpha] \tag{7.49}$$

$$e^w\left(t^i\right) = \sum_{a \epsilon B(s)} \sum_p v_{ap}^w\left(t^i\right) \qquad \forall i, w \tag{7.50}$$

$$\sum_{a \epsilon A(n)} u_{ap}^w\left(t^i\right) = \sum_{a \epsilon B(n)} v_{ap}^w\left(t^i\right) \qquad \forall i, w, p \in P_w, n \quad [\gamma] \tag{7.51}$$

$$\sum_p f_p^w\left(t^i\right) = d^w\left(t^i\right) \qquad \forall i, w \quad [\delta] \tag{7.52}$$

$$\sum_j y_{t^j t^i}^w = d^w\left(t^i\right) \qquad \forall i, w \quad [\delta] \tag{7.53}$$

$$\sum_i y_{t^j t^i}^w = \hat{d^w}\left(t^j\right) \qquad \forall j, w \quad [\zeta] \tag{7.54}$$

$$\sum_i d^w\left(t^i\right) = \sum_i \hat{d^w}\left(t^i\right) \qquad \forall w \quad [\varsigma] \tag{7.55}$$

$d^w(t^i)$ is the optimizable demand (used in the system optimization problem) for the w^{th} OD pair during the i^{th} departure time interval. $\hat{d^w}\left(t^j\right)$ is the observed demand pattern (as read from the input time-dependent OD matrix) for the w^{th} OD pair during the j^{th} departure time interval. Note that since our model (for now) focuses on optimal route toll vector $\theta\left(t^i\right)$ that induce optimal route flow vector $f\left(t^i\right)$ during the departure time t^i, Eq.(7.50) can be omitted, since Eq.(7.52) takes care of arrival flows. When optimizing the entire system flow, constraints (7.52) to (7.55) ensure the flexibility of the demand within the periods, and realisation of the total demand in the modelling period T. The Greek letters in the square brackets are the *Karush–Kuhn–Tucker (KKT) multipliers* associated with the constraints.

Definitional constraints

For all i, the following conditions hold:

$$\sum_{w,p} u_{ap}^{w}\left(t^{i}\right) = u_{a}\left(t^{i}\right), \qquad \sum_{w,p} v_{ap}^{w}\left(t^{i}\right) = v_{a}\left(t^{i}\right) \qquad \forall a \tag{7.56}$$

$$u_{bp}^{w}\left(t^{i}\right) = v_{ap}^{w}\left(t^{i}\right) \qquad \forall a \in A(r),\ A(s), b \in B(r),\ B(s) \tag{7.57}$$

$$\sum_{p} x_{ap}^{w}\left(t^{i}\right) = x_{a}^{w}\left(t^{i}\right), \qquad \sum_{w,p} x_{ap}^{w}\left(t^{i}\right) = \sum_{w} x_{a}^{w}\left(t^{i}\right) = x_{a}\left(t^{i}\right) \quad \forall w, a \tag{7.58}$$

$$\sum_{p} E_{p}^{w}\left(t^{i}\right) = E^{w}\left(t^{i}\right) \qquad \forall w \tag{7.59}$$

In condition (7.57), we have created artificial inflow links into the origin node r. This is common in traffic modelling, where artificial links are created from the centroids (commonly referred to as zones) to the physical origin and destination nodes (see Figure 7.3). Observe that equation (7.51) is well satisfied at both origin and destination nodes r and s.

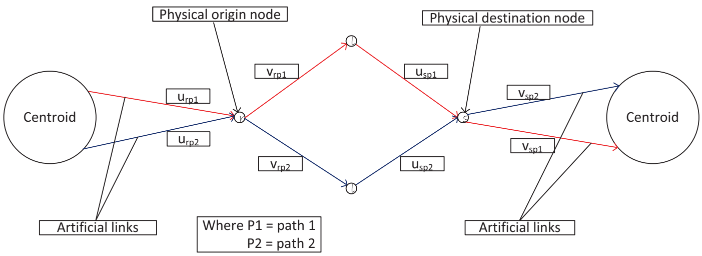


Figure 7.3: Diagrammatic explanation of *Eq.(7.57)*

Non negativity conditions

$$u_{ap}^{w}\left(t^{i}\right) \geq 0 \quad [\lambda], \quad v_{ap}^{w}\left(t^{i}\right) \geq 0 \quad [\xi], \quad x_{ap}^{w}\left(t^{i}\right) \geq 0 \quad \forall i, w, p \in P_{w}, a \tag{7.60}$$

$$E_{p}^{w}\left(t^{i}\right) \geq 0 \qquad \forall i, w, p \in P_{w} \tag{7.61}$$

$$y_{t^{i}t^{j}}^{w} \geq 0 \quad [\varrho] \qquad \forall i, j, w \tag{7.62}$$

Boundary conditions

$$E_{p}^{w}\left(t^{0}\right) = 0 \qquad \forall w, p \in P_{w} \tag{7.63}$$

$$x_{ap}^{w}\left(t^{0}\right) = 0 \qquad \forall w, p \in P_{w}, a \tag{7.64}$$

Relationships between state and control variables

$$\frac{d}{dt^{i}} x_{ap}^{w}\left(t^{i}\right) = u_{ap}^{w}\left(t^{i}\right) - v_{ap}^{w}\left(t^{i}\right) \qquad \forall a, w, p \in P_{w}, i \tag{7.65}$$

$$\frac{d}{dt^{i}} E_{p}^{w}\left(t^{i}\right) = e_{p}^{w}\left(t^{i}\right) \qquad \forall w, p \in P_{w}, i \tag{7.66}$$

Flow propagation condition

Flow propagates as in fixed demand models.

The Greek letters in the square brackets are the *Karush–Kuhn–Tucker (KKT) multipliers* associated with the constraints.

User problem (UP)

The user problem (usually) formulated at the lower level of the bi-level road pricing problem is the multi-period version of the static Wardrop's equilibrium. We define this multi-period user equilibrium (MPUE) to be the state of the traffic in which no user decreases his or her generalised travel cost by unilaterally changing routes or departure time. As demonstrated in Appendix A, it was shown that this equilibrium condition can be found by solving an equivalent variational inequality (VI) problem (see also [19, 50, 11]). Adding the transfer cost $c^w_{t^j t^i}$ (which users incur by departing at t^i instead of t^j), we reformulate the VI for the dynamic user equilibrium (DUE) in [19, 18] as follows:

Given that F is the set of all feasible path flows, then

$$Find\, \tilde{f}^w_p\left(t^i\right) \in F\, such\, that$$

$$\sum_w \sum_i \left(\sum_{p \epsilon P_w} \left(\tilde{\eta}^w_p\left(t^i\right)\right)\left(f^w_p\left(t^i\right) - \tilde{f}^w_p\left(t^i\right)\right) + \sum_j \left[c^w_{t^j t^i} \cdot z^w_{t^j t^i}\right] \right) \geq 0 \quad \forall f^w_p\left(t^i\right) \in F \tag{7.67}$$

The variational inequality above can be written as a minimization problem

Find $\tilde{f} = \tilde{f}^w_p\left(t^i\right)$ such that $\tilde{f}$ solves

$$\min_{f,z} \sum_w \sum_i \left(\sum_{p \epsilon P_w} \left[\tilde{\eta}^w_p\left(t^i\right) f^w_p\left(t^i\right)\right] + \sum_j \left[c^w_{t^j t^i} \cdot z^w_{t^j t^i}\right] \right)$$
$$s.t$$
$$f^w_p\left(t^i\right) \in F$$

where $\eta^w_p\left(f\left(t^i\right)\right)$ short-written as $\eta^w_p\left(t^i\right)$ is the flow-dependent travel time on route $p \in P_w$, $\tilde{f}^w_p\left(t^i\right)$ is the route flow on route $p \in P_w$.

Again, for a given OD pair $w \in W$, $c^w_{t^j t^i}$ is the cost involved in shifting departure time of a user from t^j to t^i [9]. $z^w_{t^j t^i}$ is the number of users who prefer to depart during departure time interval t^j but are actually departing during t^i. The tilde '$\sim$' indicates a fixed parameter. Note also that the transfer cost $c^w_{t^j t^i}$ is known.

We therefore formulate the UP as follows:

$$\min_{f,z} \sum_w \sum_i \left(\sum_{p \epsilon P_w} \left[\tilde{\eta}^w_p\left(t^i\right) f^w_p\left(t^i\right)\right] + \sum_j \left[c^w_{t^j t^i} \cdot z^w_{t^j t^i}\right] \right)$$
$$s.t$$

$$\begin{aligned}
f_p^w\left(t^i\right) &= \sum_{a \epsilon A(r)} u_{ap}^w\left(t^i\right) && \forall i, w, p \in P_w, \quad [\alpha] \\
e^w\left(t^i\right) &= \sum_{a \epsilon B(s)} \sum_{p \epsilon P_w} v_{ap}^w\left(t^i\right) && \forall i, w \\
\sum_{a \epsilon A(l)} u_{ap}^w\left(t^i\right) &= \sum_{a \epsilon B(l)} v_{ap}^w\left(t^i\right) && \forall i, w, p \in P_w, l \quad [\gamma] \\
\sum_{p \epsilon P_w} f_p^w\left(t^i\right) &= \hat{d}^w\left(t^i\right) && \forall i, w \quad [\delta] \\
\sum_j z_{t^j t^i}^w &= \hat{d}^w\left(t^i\right) && \forall i, w \quad [\delta]
\end{aligned} \tag{7.68}$$

where$\delta^w\left(t^j\right)$ is a free variable corresponding to the minimum travel cost on route $p \in P_w$ for users departing during t^j, and $\hat{d}^w\left(t^j\right)$ is the observed demand of users departing an origin r toward a destination s during time slot t^j. Further, $c_{t^j t^i}^w$ is the transfer cost from t^j to t^i, where we take that the diagonal of the matrix c_{TT}^W has zero entries, *i.e.* $c_{t^i t^i}^w = 0$ for all departure times i and all origin–destination pair w. Observe from system (7.68) that the route travel cost $\tilde{\eta}_p^w\left(t^i\right)$ is fixed in accord with the VI in Eq.(7.67). This is actually the main difference between the objective formulation of the user problem (7.68) and that of the system problem (7.70).

With the conditions in system (7.68), the relational and definitional conditions are satisfied. The boundary conditions are hard coded. The Greek letters in the square brackets are the *Karush–Kuhn–Tucker (KKT) multipliers* associated with the constraints.

We derived in Appendix A that *any flow pattern* $\tilde{f}_p^w\left(t^j\right), 1, \cdots, T$ *satisfying the following conditions is a multi-period user equilibrium (MPUE) flow pattern:*

$$\boxed{\begin{aligned}
\tilde{\eta}_p^w\left(t^i\right) + c_{t^j t^i}^w \geq \delta^w\left(t^j\right) \qquad & \forall w \in W, j \\
\sum_{p \epsilon P_w}\left(\tilde{\eta}_p^w\left(t^i\right)\right)\tilde{f}_p^w\left(t^i\right) = \delta^w\left(t^i\right)\hat{d}^w\left(t^i\right) \qquad & \forall w \in W
\end{aligned}} \tag{7.69}$$

for a given departure time slot t^i,

System problem (SP)

We assume that the controller's objective is to minimize the system's total travel cost: travel time cost and the cost of shifting users from one departure time interval to another. This objective is thus stated as follows:

$$\begin{gathered}
\underset{f,y}{min} \sum_i \left(\sum_w \sum_{p \epsilon P_w}\left[f_p^w\left(t^i\right)\eta_p^w\left(t^i\right)\right] + \sum_j \sum_w \left[c_{t^j t^i}^w \cdot y_{t^j t^i}^w\right]\right) \\
s.t \\
flow\, feasibility\, constraints \text{ (Eqs. 7.49-7.55)} \\
nonnegativity\, constraints \text{ (Eqs. 7.60-7.62)}
\end{gathered} \tag{7.70}$$

With the conditions in Eq.(7.70), the relational and definitional conditions are satisfied. The boundary conditions are hard coded.
$f_p^w(t^i)$ is the departure flow rate along route p for the w^{th}OD pair during departure time interval t^i.

For a given OD pair $w \in W$, $c^w_{t^jt^i}$ is the cost involved in shifting a user's departure time from t^j to t^i [9]. The cost-matrix c^W_{TT} is determined using the technique of reversed engineering approach based on a discrete choice model, of type logit, with a utility function adopted from [61] as described in [9]. This technique involves the use of Small's formulation of the time of travel choice problem, to determine the preferred time of departure t^H by the users given the observed departure time pattern t^J [61]. Knowing the preferred departure time t^H, one can then determine the cost involved in shifting demand from t^J to t^I, an interested reader is referred to [61, 9]. The diagonal of the matrix c^W_{TT} has zero entries, that is $c^w_{t^it^i} = 0$ for all departure times i and all origin–destination pair w.

$\eta_p^w(t^i)$ is travel time experienced over route p by users belonging to the w^{th} OD pair during t^i. Again, we have used $\eta_p^w(t^i)$ to mean $\eta_p^w(f(t^i))$. Observe that the cost $\eta_p^w(f(t^i))$ on route $p \in P_w$ depends on a whole vector of path flows $f(t^i)$, and this means that $\eta_p^w(f(t^i))$ may depend on the flows on other routes $q \in P_w$ with $q \neq p$ as well as on $p \in P_m$ with $m \neq w$. Due to *Assumption 2*, the route cost $\eta_p^w(f(t^i))$ is a continuously differentiable function of the route flow $f_p^w(t^i)$.

For the OD pair $w \in W$, $y^w_{t^jt^i}$ is the number of users who in the observed travel pattern $\hat{d}$ were departing during departure time interval t^j, and will be departing in the interval t^i in the optimized pattern d.

In Eq.(7.70), the first part of the objective minimizes the system travel time cost, and the second part minimizes the system cost involved in shifting the departure times of users from t^j to t^i, also called the transfer cost. Note that the choice of travel time cost $f_p^w(t^i)\,\eta_p^w(t^i)$ is an arbitrary choice. The system controller instead can minimize the cost of emission, noise, etcetera or any combination of the cost as deemed fit.

Suppose $\left(\bar{f}, \bar{d}, \bar{y}\right)$ solves system (7.70), we derived in Appendix A that *for any departure time interval t^i, any route toll $\theta_p^w(t^i)$, with $p \in P_w$, satisfying the following linear conditions will also induce the optimal feasible route flow pattern $\bar{f}_p^w(t^j)$ as a multi-period user equilibrium (MPUE) flow pattern:*

$$\boxed{\begin{aligned} &\left(\bar{\eta}_p^w\left(t^i\right) + c^w_{t^jt^i} + \theta_p^w\left(t^i\right)\right) \geq \zeta^w\left(t^j\right) && \forall p \in P_w, w \in W, j \\ &\sum_{p \epsilon P_w} \left(\bar{\eta}_p^w\left(t^i\right) + \theta_p^w\left(t^i\right)\right)\bar{f}_p^w\left(t^i\right) = \zeta^w\left(t^i\right)\bar{d}^w\left(t^i\right) && \forall w \in W \end{aligned}} \tag{7.71}$$

where $\zeta^w(t^j)$ is a free variable corresponding to the minimum travel cost on route $p \in P_w$ for users departing during t^j, and $\bar{d}^w(t^j)$ is the optimal demand of users departing an origin r toward a destination s during t^j. We have also used $\bar{\eta}_p^w(t^i)$ to mean $\eta_p^w\left(\bar{f}(t^i)\right)$. Note that the optimized demand $\bar{d}^w(t^j)$ need not to be the same as the observed demand $\hat{d}^w(t^j)$ for every j, but $\sum_j \bar{d}^w(t^j) = \sum_j \hat{d}^w(t^j)$ as explained in subsection 7.5.1.

For an OD-based pricing, we will let the route tolls $\theta_p^w(t^i)$ in (7.71) be identical for all $p \in P_w$. With the first-best solution $\left(\bar{f}, \bar{d}, \bar{y}\right)$ from system (7.70), one first solves the linear system in Eq.(7.71) to see if an OD-based pricing scheme exists which can induce $\left(\bar{f}, \bar{d}, \bar{y}\right)$ as an MPUE. If such a scheme exists, then the OD-based scheme yields the system optimal (first-best) solution as in subsection 7.4.1, otherwise, using $\left(\bar{f}, \bar{d}, \bar{y}\right)$ as an initial solution point, one has to solve the following problem:

$$\begin{gathered}
\min_{f,y,\theta} \sum_i \left(\sum_w \sum_p \left[f_p^w \left(t^i\right) \eta_p^w \left(t^i\right) \right] + \sum_j \sum_w \left[c_{t^j t^i}^w \cdot y_{t^j t^i}^w \right] \right) \\
s.t \\
flow\, feasibility\, \&\, nonnegativity\, constraints \\
\left(\eta_p^w \left(t^i\right) + c_{t^j t^i}^w + \theta^w \left(t^i\right) \right) \geq \zeta^w \left(t^j\right) \qquad \forall p \in P_w, w, i, j \\
\sum_{p \epsilon P_w} \left(\eta_p^w \left(t^i\right) + \theta^w \left(t^i\right) \right) f_p^w \left(t^i\right) = \zeta^w \left(t^i\right) d^w \left(t^i\right) \qquad \forall w, i
\end{gathered} \tag{7.72}$$

The second and the third conditions ensure that the resulting feasible flow is in multi-period user equilibrium (MPUE). $\theta^w\left(t^i\right)$ is the modelling OD-based toll vector that enables the optimisation of the path flows $f_p^w\left(t^i\right)$and the demand shifts $d^w\left(t^i\right)$, and further ensures the multi-period user equilibrium state of the optimised flow pattern f: a state in which no user thinks he or she can decrease his or her generalised travel cost by unilaterally changing routes or departure time (see Appendix A for more explanations). All other variables in system (7.72) remain as previously described.

7.5.3 The second-best OD-based pricing for a multi-period static traffic assignment

Here we define the second-best scheme to mean a tolling scheme where tolls are not allowed on paths connecting a given OD pair $w \in W$. This requirement may just be for a given time interval. Therefore, for a given origin–destination pair *w,* all paths $p \in P_w$ may be required to be toll free during the interval t^i, and may take a positive toll during t^j, where $i \neq j$. If we denote by $\omega(t^i)$ the set of all toll free OD pairs during departure interval t^i, then it is required that

$$\theta^w \left(t^i\right) = 0\, \forall w \in \omega(t^i) \tag{7.73}$$

If condition (7.73) is required, then one only need to add this extra condition (Eq.(7.73)) to system (7.72).

7.6 Numerical example for the MSTA

We will demonstrate the OD-based pricing model for the multi-period static traffic assignment (MSTA) using the Nguyen and Dupuis network (see Figure 7.4). The network has 13 nodes, 19 links, 25 paths and 4 origin–destination pairs. In this example, we have chosen to minimize the system travel time cost $f^T\eta$ and the transfer cost $c^T y$. We suppose that the time window T is divided into three discrete departure time intervals t^i with i = *1, 2, 3.* We further suppose undifferentiated users. The figure and the table below give the example network, and the link attributes respectively.

7.6.1 The Nguyen and Dupuis network example

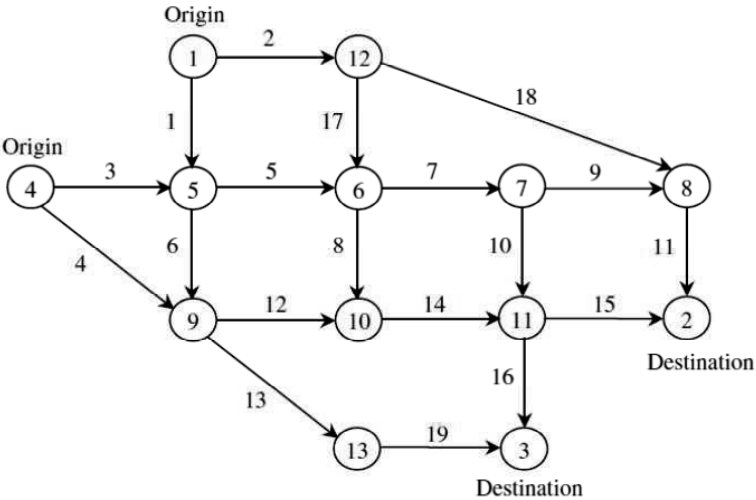


Figure 7.4: The Nguyen and Dupuis network with node and link numbers

Table 7.5: Link attributes

Link Identity Number	Link Capacity	FreeFlowTravelTime (mins)
1	700	7
2	560	9
3	560	12
4	375	5
5	420	12
6	420	9
7	700	5
8	280	4
9	700	9
10	700	4
11	280	9
12	280	5
13	280	9
14	700	4
15	280	9
16	560	8
17	140	7
18	560	18
19	560	11

OD	p	Route
I 1–>2	1	1 5 6 7 8 2
	2	1 5 6 7 11 2
	3	1 5 6 10 11 2
	4	1 5 9 10 11 2
	5	1 12 8 2
	6	1 12 6 7 8 2
	7	1 12 6 7 11 2
	8	1 12 6 10 11 2
II 1–>3	9	1 5 6 7 11 3
	10	1 5 6 10 11 3
	11	1 5 9 10 11 3
	12	1 5 9 13 3
	13	1 12 6 7 11 3
	14	1 12 6 10 11 3
III 4–>2	15	4 5 6 7 8 2
	16	4 5 6 7 11 2
	17	4 5 6 10 11 2
	18	4 5 9 10 11 2
	19	4 9 10 11 2
IV 4–>3	20	4 5 6 7 11 3
	21	4 5 6 10 11 3
	22	4 5 9 10 11 3
	23	4 5 9 13 3
	24	4 9 10 11 3
	25	4 9 13 3

Table 7.6: Transfer cost per OD

Transfer cost C_{TT}^{I}

	t^1	t^2	t^3
t^1	0.00	3.07	0.22
t^2	3.07	0.00	2.84
t^3	0.22	2.84	0.00

Transfer cost C_{TT}^{II}

	t^1	t^2	t^3
t^1	0.00	0.71	0.01
t^2	0.71	0.00	0.72
t^3	0.01	0.72	0.00

Transfer cost C_{TT}^{III}

	t^1	t^2	t^3
t^1	0.00	6.19	0.11
t^2	6.19	0.00	6.08
t^3	0.11	6.08	0.00

Transfer cost C_{TT}^{IV}

	t^1	t^2	t^3
t^1	0.00	2.66	0.05
t^2	2.66	0.00	2.71
t^3	0.05	2.71	0.00

Table 7.7: Observed demand (Input OD demand matrix)

Time slot	OD	Demand
t^1	I	200
	II	400
	III	200
	IV	150
t^2	I	500
	II	700
	III	550
	IV	250
t^3	I	350
	II	300
	III	200
	IV	100

Table 7.5 gives the link characteristics of the eight links. Table 7.6 gives the transfer costs $c_{t^j t^i}$ involved in shifting departure time of a user from t^j to t^i [9]. Table 7.7 gives the observed daily traffic pattern for the example network and for the three multiple periods t^1, t^2 and t^3.

We use the so called *Bureau for Public Roads (BPR) function* $\alpha T_a^{ff}\left(1+\varphi\left(\frac{v_a(t^i)}{\hat{C}_a}\right)^{\phi}\right)$ to define the link travel time cost where
T_a^{ff} – free flow travel time on link a,
$v_a(t^i)$ – total flow on link a during t^i.
$\hat{C}_a$ – practical capacity of link a, and
$\varphi\, and\, \phi$ – BPR scaling parameters, with $\varphi = 0.15$, $\phi = 4$.
α is the value of time (VOT) with the value *0.1671667 / minute* [4]. In reality, it could be that some travellers have no choice than to strictly depart during a given departure time t^i. Therefore, we assume for each OD, that the following number of travellers is bound to depart during the corresponding departure intervals:

Table 7.8: Fixed demand

OD/departure time	t^1	t^2	t^3
I	200	300	250
II	150	500	230
III	200	350	150
IV	50	80	150

Other users are flexible with respect to departure times, but a cost is incurred in shifting them from one departure time interval to another.

7.6.2 Results

Tolls and costs are in Euro (€).

Table 7.9: User equilibrium: observed traffic scenario for the three discrete time intervals

Table 7.9a: Observed table

t^1

OD $[w]$	Paths $[p]$	Path flows $[f_p]$	Path tolls $[\theta_p]$	Path Cost $[\eta_p]$	System Cost $[f_p\eta_p]$
I	1	0	0.00	7.08	0.00
	2	1	0.00	6.13	5.71
	3	0	0.00	6.21	0.00
	4	0	0.00	6.21	0.00
	5	197	0.00	6.13	1,205.05
	6	0	0.00	7.08	0.00
	7	2	0.00	6.13	14.28
	8	0	0.00	6.21	0.00
	Demand	**200**		**Total cost**	**1,225.04**
II	9	0	0.00	6.09	0.00
	10	76	0.00	6.03	455.30
	11	86	0.00	6.03	515.60
	12	63	0.00	6.03	382.09
	13	5	0.00	6.03	27.23
	14	171	0.00	6.03	1,031.96
	Demand	**400**		**Total cost**	**2,412.19**
III	15	0	0.00	7.91	0.00
	16	0	0.00	7.08	0.00
	17	0	0.00	7.04	0.00
	18	0	0.00	7.05	0.00
	19	200	0.00	4.46	892.99
	Demand	**200**		**Total cost**	**892.99**
IV	20	0	0.00	6.92	0.00
	21	0	0.00	6.88	0.00
	22	0	0.00	6.89	0.00
	23	0	0.00	6.87	0.00
	24	89	0.00	4.28	379.89
	25	61	0.00	4.28	262.35
	Demand	**150**		**Total cost**	**642.23**
Total					**5,172.45**

Table 7.9b: Observed table

t^2

OD [w]	Paths [p]	Path flows [f_p]	Path tolls [θ_p]	Path Cost [η_p]	System Cost [$f_p\eta_p$]
I	1	0	0.00	9.89	0.00
	2	0	0.00	9.89	0.00
	3	0	0.00	9.89	0.00
	4	0	0.00	10.46	0.00
	5	500	0.00	9.19	4,596.51
	6	0	0.00	9.89	0.00
	7	0	0.00	9.89	0.00
	8	0	0.00	9.90	0.00
	Demand	**500**		**Total cost**	**4,596.51**
II	9	212	0.00	6.74	1,426.08
	10	199	0.00	6.74	1,337.89
	11	0	0.00	7.31	0.00
	12	111	0.00	6.74	746.99
	13	134	0.00	6.74	901.95
	14	45	0.00	6.74	305.79
	Demand	**700**		**Total cost**	**4,718.71**
III	15	6	0.00	10.66	60.08
	16	34	0.00	10.66	359.78
	17	3	0.00	10.66	26.73
	18	0	0.00	11.24	0.00
	19	508	0.00	10.66	5,415.40
	Demand	**550**		**Total cost**	**5,861.99**
IV	20	0	0.00	7.52	0.00
	21	0	0.00	7.53	0.00
	22	0	0.00	8.09	0.00
	23	0	0.00	7.52	0.00
	24	0	0.00	7.51	0.00
	25	250	0.00	6.94	1,735.73
	Demand	**250**		**Total cost**	**1,735.73**
Total					**16,912.94**

Table 7.9c: Observed table

t^3

OD $[w]$	Paths $[p]$	Path flows $[f_p]$	Path tolls $[\theta_p]$	Path Cost $[\eta_p]$	System Cost $[f_p \eta_p]$
I	1	0	0.00	7.25	0.00
	2	0	0.00	6.39	0.00
	3	19	0.00	6.35	120.98
	4	12	0.00	6.35	73.46
	5	279	0.00	6.35	1,771.28
	6	0	0.00	7.23	0.00
	7	6	0.00	6.35	35.71
	8	35	0.00	6.35	221.05
	Demand	**350**		**Total cost**	**2,222.48**
II	9	7	0.00	6.03	43.78
	10	76	0.00	6.03	455.70
	11	86	0.00	6.03	518.80
	12	0	0.00	6.04	0.00
	13	0	0.00	6.05	0.00
	14	131	0.00	6.03	790.22
	Demand	**300**		**Total cost**	**1,808.50**
III	15	0	0.00	8.08	0.00
	16	0	0.00	7.22	0.00
	17	0	0.00	7.20	0.00
	18	0	0.00	7.20	0.00
	19	200	0.00	4.58	915.04
	Demand	**200**		**Total cost**	**915.04**
IV	20	0	0.00	6.90	0.00
	21	0	0.00	6.87	0.00
	22	0	0.00	6.88	0.00
	23	0	0.00	6.85	0.00
	24	79	0.00	4.23	335.61
	25	21	0.00	4.23	87.40
	Demand	**100**		**Total cost**	**423.01**
				Total	**5,369.03**

As a reference point, we solve the uncontrolled user problem. This describes the traffic situation without tolling, and the results are given as an observed traffic scenario in Table 7.9.

Given that all ODs can be tolled, we now solve the system problem (system (7.72)) for the optimal OD tolls $\theta^w(t^i)$ and the corresponding path flows $f_p^w(t^i)$. The OD tolls $\theta^w(t^i)$ as given in Table 7.10 ensure that the demand is efficiently distributed through the modelling period given the transfer costs. The OD tolls further ensure multi-period user equilibrium in the network.

Observe from Tables *7.9* and 7.10 that the traffic flow patterns obey the multi-period Wardrop's equilibrium, where for every OD and a given departure time interval t^i, the costs on all used paths are the same, and smaller than the costs on the unused paths, with no user having any incentive to switch paths or departure time.

For this simple network, the OD-based toll improved the no-toll scenario by about 10% (see table 7.11). Though we are not optimizing over the tolls, it is interesting to see that (despite the fact that it is just a mere coincidence) the OD-dependent tolls required to achieve the desired path flows are very small for all departure time intervals. It is important to recall that the (OD) toll patterns as given in Table 7.10 are in general not unique. In fact, there exist infinite (OD) toll patterns that can achieve this same (path) flow pattern (see the appendix).

Table 7.10: OD-based toll: optimised traffic scenario for the three discrete time intervals

Table 7.10a: Optimised table

t^1

OD [w]	Paths [p]	Path flows [f_p]	Path tolls [θ_p]	Path Cost [η_p]	System Cost [$f_p\eta_p$]
I	1	0	1.33	8.43	0.00
	2	0	1.33	7.88	0.00
	3	0	1.33	7.84	0.00
	4	0	1.33	7.84	0.00
	5	200	1.33	7.47	1,227.08
	6	0	1.33	8.42	0.00
	7	0	1.33	7.87	0.00
	8	0	1.33	7.83	0.00
	Demand	**200**		**Total cost**	**1,227.08**
II	9	77	1.85	7.90	468.51
	10	102	1.85	7.90	618.86
	11	24	1.85	7.90	147.56
	12	79	1.85	7.90	480.49
	13	31	1.85	7.90	185.55
	14	147	1.85	7.90	892.01
	Demand	**461**		**Total cost**	**2,792.98**
III	15	0	0.35	8.27	0.00
	16	0	0.35	7.72	0.00
	17	0	0.35	7.68	0.00
	18	0	0.35	7.68	0.00
	19	311	0.35	5.25	1,528.08
	Demand	**311**		**Total cost**	**1,528.08**
IV	20	0	0.86	7.79	0.32
	21	0	0.86	7.75	0.00
	22	0	0.86	7.74	0.05
	23	0	0.86	7.75	0.06
	24	40	0.86	5.32	177.31
	25	93	0.86	5.32	413.59
	Demand	**133**		**Total cost**	**591.32**
				Total	**6,139.46**

Table 7.10b: Optimised table

t^2

OD $[w]$	Paths $[p]$	Path flows $[f_p]$	Path tolls $[\theta_p]$	Path Cost $[\eta_p]$	System Cost $[f_p\eta_p]$
I	1	0	0.83	8.87	0.00
	2	0	0.83	8.15	0.00
	3	0	0.83	8.14	0.00
	4	0	0.83	8.20	0.00
	5	396	0.83	8.11	2,884.01
	6	0	0.83	8.88	0.00
	7	0	0.83	8.15	0.00
	8	0	0.83	8.14	0.00
	Demand	**396**		**Total cost**	**2,884.01**
II	9	127	1.09	7.31	788.52
	10	184	1.09	7.31	1,143.26
	11	0	1.09	7.38	0.00
	12	107	1.09	7.31	665.96
	13	94	1.09	7.31	584.15
	14	76	1.09	7.31	472.28
	Demand	**587**		**Total cost**	**3,654.18**
III	15	0	0.19	9.04	0.00
	16	0	0.19	8.32	0.00
	17	0	0.19	8.30	0.00
	18	0	0.19	8.37	0.00
	19	408	0.19	6.33	2,508.52
	Demand	**408**		**Total cost**	**2,508.52**
IV	20	0	0.44	7.49	0.00
	21	0	0.44	7.48	0.00
	22	0	0.44	7.55	0.00
	23	0	0.44	7.48	0.00
	24	0	0.44	5.51	0.00
	25	155	0.44	5.44	776.02
	Demand	**155**		**Total cost**	**776.02**
				Total	**9,822.74**

Table 7.10c: Optimised table

t^3

OD $[w]$	Paths $[p]$	Path flows $[f_p]$	Path tolls $[\theta_p]$	Path Cost $[\eta_p]$	System Cost $[f_p\eta_p]$
I	1	0	0.98	8.50	0.24
	2	48	0.98	7.67	322.55
	3	35	0.98	7.67	232.80
	4	1	0.98	7.67	5.38
	5	337	0.98	7.67	2,259.12
	6	0	0.98	8.49	1.24
	7	18	0.98	7.67	118.59
	8	14	0.98	7.67	95.62
	Demand	**453**		**Total cost**	**3,035.54**
II	9	35	1.84	7.90	211.22
	10	82	1.84	7.90	499.17
	11	52	1.84	7.90	314.74
	12	47	1.84	7.90	281.87
	13	17	1.84	7.90	105.41
	14	119	1.84	7.90	718.14
	Demand	**352**		**Total cost**	**2,130.54**
III	15	0	0.05	8.40	0.00
	16	0	0.05	7.61	0.00
	17	0	0.05	7.58	0.00
	18	0	0.05	7.58	0.00
	19	231	0.05	5.14	1,174.63
	Demand	**231**		**Total cost**	**1,174.63**
IV	20	0	0.92	7.84	0.11
	21	0	0.92	7.80	0.04
	22	0	0.92	7.80	0.01
	23	0	0.92	7.80	0.03
	24	94	0.92	5.37	418.90
	25	118	0.92	5.37	526.27
	Demand	**212**		**Total cost**	**945.37**
				Total	**7,286.08**

Table 7.10d: Transfers and associated costs

Transfers (y^{I}_{TT})

	t^1	t^2	t^3
t^1	200	0	0
t^2	0	396	104
t^3	0	0	350

Total transfer cost ($\sum_j y^{I}_{ji} C^{I}_{ji}$) to:

t^1	0.00
t^2	0.00
t^3	294.42

Transfers (y^{II}_{TT})

	t^1	t^2	t^3
t^1	400	0	0
t^2	61	587	52
t^3	0	0	300

Total transfer cost ($\sum_j y^{II}_{ji} C^{II}_{ji}$) to:

t^1	43.39
t^2	0.00
t^3	37.40

Transfers (y^{III}_{TT})

	t^1	t^2	t^3
t^1	200	0	0
t^2	111	408	31
t^3	0	0	200

Total transfer cost ($\sum_j y^{III}_{ji} C^{III}_{ji}$) to:

t^1	690.24
t^2	0.00
t^3	185.72

Transfers (y^{IV}_{TT})

	t^1	t^2	t^3
t^1	133	0	17
t^2	0	155	95
t^3	0	0	100

Total transfer cost ($\sum_j y^{IV}_{ji} C^{IV}_{ji}$) to:

t^1	0.00
t^2	0.00
t^3	258.31

Table 7.11: Summary table

	Observed	Optimized
Total demand		
ODI	1,050	1,050
ODII	1,400	1,400
ODIII	950	950
ODIV	500	500
System costs		
Total travel time cost (€)	27,454.42	23,248.28
Total transfer cost (€)	0.00	1,509.48
Total system cost (€)	27,454.42	24,757.76
Cost reduction due to OD toll (€)		2,696.66
Percentage cost reduction (%)		9.82

7.7 Conclusions

Due to some practical issues arising from the link-based (or route-based) pricing schemes, origin–destination based (OD-based) road pricing presents a potential tool to alleviate these issues. In this Chapter, we study this new pricing scheme, and the contributions of the Chapter to the literature are: (1) the analytical derivation of OD-based tolls for elastic demand under the static traffic assignment model, (2) the derivation of equilibrium conditions for a multi-period static traffic assignment (MSTA), and (3) the formulation of an OD-based pricing scheme for the MSTA. The OD tolls for elastic demand regulate the overall demand effect, which is the extent to which road users efficiently leave or enter the road system due to congestion pricing. Further, the OD-based pricing scheme for a multi-period static traffic assignment regulates the overall shift demand effect, which is the extent to which road users efficiently choose a given departure time due to congestion pricing, and the costs associated with shifting departure times. Numerical examples show that the OD-based tolling scheme has potentials of greatly improving the network and reduce the total travel cost. We acknowledge that the proposed scheme has a downside of not being able to optimize the route split as in the first-best link-based pricing scheme. Our next line of research will be to explicitly derive the OD-based tolls for the MSTA in a closed form.

Appendix A

Multi-period static traffic assignment (MSTA)

Derivation of Eqs.(7.69) and (7.71).

System problem (SP)

Given the feasibility conditions in Section 5.2.1 and SP formulation in system (7.70), we derive the following:

If we let L be the Lagrangian, and $\left(\bar{f}_p^w\left(t^i\right), \bar{y}_{t^j t^i}^w\right)$ (with the corresponding path cost $\bar{\eta}_p^w\left(t^i\right)$) be the solution to program (7.70), then, for a given t^i, there exists $(\alpha, \gamma, \delta, \zeta, \varsigma, \lambda, \xi, \varrho)$ such that the following *KKT conditions* hold:

$$\begin{aligned}
L \;=\; & \sum_i \sum_w \sum_{p \epsilon P_w} \left[f_p^w\left(t^i\right) \eta_p^w\left(t^i\right)\right] + \sum_i \sum_j \sum_w \left[c_{t^j t^i}^w \cdot y_{t^j t^i}^w\right] + \left(\sum_{a \epsilon A(r)} u_{ap}^w\left(t^i\right) - f_p^w\left(t^i\right)\right) \alpha \\
& + \left(\sum_{a \epsilon B(n)} v_{ap}^w\left(t^i\right) - \sum_{a \epsilon A(n)} u_{ap}^w\left(t^i\right)\right) \gamma + \left(d^w\left(t^i\right) - \sum_p f_p^w\left(t^i\right)\right) \delta \\
& + \left(\sum_j y_{t^j t^i} - d^w\left(t^i\right)\right) \delta + \left(\hat{d^w}\left(t^j\right) - \sum_i y_{t^j t^i}\right) \zeta + \left(\sum_i \hat{d^w}\left(t^i\right) - \sum_i d^w\left(t^i\right)\right) \varsigma \\
& - u_{ap}^w\left(t^i\right) \lambda - v_{ap}^w\left(t^i\right) \xi - y_{t^j t^i}^w \varrho
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial f_p^w\left(t^i\right)} L &= \left(\bar{\eta}_p^w\left(t^i\right) + \bar{f}_p^w\left(t^i\right) \frac{d}{d f_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)\right) - \alpha_p^w\left(t^i\right) - \delta^w\left(t^i\right) \\
&= 0 \quad \forall w, p \in P_w && (7.74) \\
\frac{\partial}{\partial u_{ap}^w\left(t^i\right)} L &= \alpha_p^w\left(t^i\right) - \gamma_p^w\left(t^i\right) - \lambda_{ap}^w\left(t^i\right) = 0 \qquad \forall w, p \in P_w, a \in p && (7.75) \\
\frac{\partial}{\partial v_{ap}^w\left(t^i\right)} L &= \gamma_p^w\left(t^i\right) - \xi_{ap}^w\left(t^i\right) = 0 \qquad \forall w, p \in P_w, a \in p && (7.76) \\
\frac{\partial}{\partial\left(d^w\left(t^i\right)\right)} L &= \delta^w\left(t^i\right) - \delta^w\left(t^i\right) - \varsigma^w\left(t^i\right) = 0 \qquad \forall w && (7.77) \\
\frac{\partial}{\partial y_{t^j t^i}^w} L &= c_{t^j t^i}^w + \delta^w\left(t^i\right) - \zeta^w\left(t^j\right) - \varrho_{t^j t^i}^w = 0 \qquad \forall j, w && (7.78) \\
u_{ap}^w\left(t^i\right) \lambda_{ap}^w\left(t^i\right) &= v_{ap}^w\left(t^i\right) \xi_{ap}^w\left(t^i\right) = 0 \qquad \forall w, p \in P_w, a \in p && (7.79) \\
y_{t^j t^i}^w \varrho_{t^j t^i}^w &= 0 \qquad \forall j, w && (7.80) \\
\lambda_{ap}^w\left(t^i\right), \xi_{ap}^w\left(t^i\right) &\geq 0 \qquad \forall w, p \in P_w, a \in p; \qquad \varrho_{t^j t^i}^w \geq 0 \qquad \forall j, w && (7.81)
\end{aligned}$$

Eqs.(7.79) and (7.80) are complementarity conditions. We have used $a \in p$ to mean that link a belongs to set of links forming path p.

From Eq.(7.74)

$$\begin{aligned}
\left(\bar{\eta}_p^w\left(t^i\right) + \bar{f}_p^w\left(t^i\right) \frac{d}{d f_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)\right) &= \alpha_p^w\left(t^i\right) + \delta^w\left(t^i\right) \\
&= \gamma_p^w\left(t^i\right) + \lambda_{ap}^w\left(t^i\right) + \delta^w\left(t^i\right) \quad \text{(due to Eq. 7.75)} \\
&= \xi_p^w\left(t^i\right) + \lambda_{ap}^w\left(t^i\right) - c_{t^j t^i}^w + \zeta^w\left(t^j\right) \\
&\quad + \varrho_{t^j t^i}^w \quad \text{(due to Eqs. 7.76 and 7.78)} \\
\left(\bar{\eta}_p^w\left(t^i\right) + \bar{f}_p^w\left(t^i\right) \frac{d}{d f_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)\right) + c_{t^j t^i}^w &\geq \zeta^w\left(t^j\right) \qquad \text{(due to Eq. 7.81)}
\end{aligned}$$

Thus, we have

$$\boxed{\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\geq\zeta^w\left(t^j\right)\qquad\forall w,j}\tag{7.82}$$

for any t^i.
Again, from Eq.(7.74)

$$\begin{aligned}
\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)\right) &= \alpha_p^w\left(t^i\right)+\delta^w\left(t^i\right)\\
&= \gamma_p^w\left(t^i\right)+\lambda_{ap}^w\left(t^i\right)+\delta^w\left(t^i\right)\ \text{(due to 7.75)}\\
&= \xi_{ap}^w\left(t^i\right)+\lambda_{ap}^w\left(t^i\right)+\delta^w\left(t^i\right)\ \text{(due to 7.76)}\\
\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right) &= \xi_{ap}^w\left(t^i\right)+\lambda_{ap}^w\left(t^i\right)+\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\ \text{(due to 7.78)}\\
\sum_{a\epsilon A(r)}\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\bar{u}_{ap}^w\left(t^j\right) &= \sum_{a\epsilon A(r)}\left(\xi_{ap}^w\left(t^i\right)+\lambda_{ap}^w\left(t^i\right)+\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{u}_{ap}^w\left(t^j\right)\\
&= \sum_{a\epsilon A(r)}\left(\xi_{ap}^w\left(t^i\right)+\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{u}_{ap}^w\left(t^j\right)\ \text{(due to 7.79)}\\
&= \sum_{a\epsilon A(r)}\left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{u}_{ap}^w\left(t^j\right)\ \text{(due to 7.57 and 7.79)}
\end{aligned}$$

$$\begin{aligned}
\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\sum_{a\epsilon A(r)}\bar{u}_{ap}^w\left(t^j\right) &= \left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\sum_{a\epsilon A(r)}\bar{u}_{ap}^w\left(t^j\right)\\
\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\bar{f}_p^w\left(t^j\right) &= \left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{f}_p^w\left(t^j\right)\\
\sum_{p\epsilon P_w}\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\bar{f}_p^w\left(t^j\right) &= \sum_{p\epsilon P_w}\left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{f}_p^w\left(t^j\right)\\
&= \left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\sum_{p\epsilon P_w}\bar{f}_p^w\left(t^j\right)\\
&= \left(\zeta^w\left(t^j\right)+\varrho_{t^jt^i}^w\right)\bar{d}^w\left(t^j\right)\quad(7.83)\\
&\qquad\text{(due to Eq. 7.52)}
\end{aligned}$$

but from Eq.(7.53),

$$\sum_i y_{t^jt^i}^w = d^w\left(t^j\right)$$

and if we let $i=j$, then we have in Eq.(7.83) that

$$\varrho_{t^jt^j}^w d^w\left(t^j\right)\ =\sum_i\ \varrho_{t^jt^j}^w y_{t^jt^j}^w=0;\quad(using\ \varrho_{t^jt^j}^w y_{t^jt^j}^w=0\qquad\forall j,w,\text{see Eq. 7.80})$$

hence, the whole of Eq.(7.83) reduces to

$$\boxed{\sum_{p\epsilon P_w}\left(\bar{\eta}_p^w\left(t^j\right)+\bar{f}_p^w\left(t^j\right)\frac{d}{df_p^w\left(t^j\right)}\left(\bar{\eta}_p^w\left(t^j\right)\right)\right)\bar{f}_p^w\left(t^j\right)=\zeta^w\left(t^j\right)\bar{d}^w\left(t^j\right)\qquad\forall w}\tag{7.84}$$

for any t^j. We have also used the fact that $c_{t^jt^j}^w=0\,\forall j$.

Therefore, we summarize as follows: f*or any given t^i, the following condition holds for the system problem*:

$$\boxed{\begin{array}{ll} \left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)+c_{t^jt^i}^w\right)\geq\zeta^w\left(t^j\right) & \forall w,j \\ \sum_{p\epsilon P_w}\left(\bar{\eta}_p^w\left(t^i\right)+\bar{f}_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\bar{\eta}_p^w\left(t^i\right)\right)\right)\bar{f}_p^w\left(t^i\right)=\zeta^w\left(t^i\right)\bar{d}^w\left(t^i\right) & \forall w \end{array}} \quad (7.85)$$

User problem (UP)

Given that $\left(\tilde{f}_p^w\left(t^i\right),\tilde{z}_{t^jt^i}^w\right)$ solves the user problem (7.68), then for a given t^i, analysing the *KKT optimality conditions* as we did in the system problem yields the following results:

$$\boxed{\tilde{\eta}_p^w\left(t^i\right)+c_{t^jt^i}^w=\xi_{ap}^w\left(t^i\right)+\lambda_{ap}^w\left(t^i\right)+\delta^w\left(t^j\right) \qquad \forall p\in P_w, w\in W, j} \quad (7.86)$$

If for route p, the inflow during time t^i into p is positive, that is, $\tilde{f}_p^w\left(t^i\right)>0$, then from Eq.(7.57), it means that $u_{bp}^w\left(t^i\right)=v_{ap}^w\left(t^i\right)>0 \quad \forall a\in A(r), b\in B(r)$. Consequently, the complementarity conditions in Eq.(7.79) force the variables $\xi_{ap}^w\left(t^i\right)$ and $\lambda_{ap}^w\left(t^i\right)$ in Eq.(7.86) to be zero. Thus, we have the following:

$$\boxed{\tilde{\eta}_p^w\left(t^i\right)+c_{t^jt^i}^w=\delta^w\left(t^j\right) \qquad \forall\tilde{f}_p^w\left(t^i\right)>0, p\in P_w, w\in W, j} \quad (7.87)$$

for any t^i.

Recall that $c_{t^it^i}^w=0\ \forall i,w$, therefore for $j=i$, Eq.(7.87) reduces to

$$\boxed{\tilde{\eta}_p^w\left(t^i\right)=\delta^w\left(t^i\right) \qquad \forall\tilde{f}_p^w\left(t^i\right)>0, p\in P_w, w\in W,\ i} \quad (7.88)$$

The LHS of (7.87) is the total equilibrated cost of traversing OD pair $w\in W$ using route $p\in P_w$, for users departing origin r towards destination s during time t^i. Observe that the RHS is a variable that does not depend on p.

Recall that $\tilde{\eta}_p^w\left(t^i\right)$ is the travel cost on route $p\in P_w$.

Interpretation: At equilibrium, the travel costs on all used routes for a given OD pair $w\in W$ are the same and equal to $\delta^w\left(t^i\right)$ for all users departing during time t^i.

For any t^i, the following holds in general due to Eq.(7.81):

$$\boxed{\tilde{\eta}_p^w\left(t^i\right)+c_{t^jt^i}^w\geq\delta^w\left(t^j\right) \qquad \forall w\in W, j} \quad (7.89)$$

Interpretation: At equilibrium, no user has an incentive to switch routes or departure time. To see this, (1) *that no user has an incentive to switch departure time*: recall from Eq.(7.88) that $\tilde{\eta}_p^w\left(t^i\right)$ is the (minimum or the actual) travel time cost of all users departing during t^i, and $c_{t^jt^i}^w$ is the transfer cost of a user who would like to switch his departure time from t^j to t^i, and from Eq.(7.88) again, we know that $\delta^w(t^j)$ is the minimum travel cost for users departing during t^j, but then, condition (7.89) states that the transfer cost from t^j to t^i, plus the minimum travel cost already experienced by users departing during t^i, is at least the minimum/actual travel cost for users departing during t^j. So at equilibrium, no user departing during t^j will have any incentive to switch departure time (to t^i).

(2) *That no user has an incentive to switch routes*: now take $j = i$ in Eq.(7.89), we then have the following:

$$\boxed{\tilde{\eta}_p^w\left(t^i\right) \geq \delta^w\left(t^i\right) \qquad \forall w \in W,\, i} \tag{7.90}$$

again $\tilde{\eta}_p^w(t^i)$ is the actual travel time cost experienced on route p for users departing during t^i. This means that at equilibrium, $\delta^w(t^i)$ must be the least travel cost between the OD pair $w \in W$ of users departing the origin during time t^i. Recall that from (7.88) $\delta^w(t^i)$ is the travel cost of all used paths. We thus state the following: *at equilibrium, the journey cost on all used paths/routes for a given OD pair are the same and equal to $\delta^w(t^i)$, but also less than those which would be experienced by a single vehicle on any of the unused paths (Wardrop's first principle).* This means that at equilibrium, no user has any incentive of switching routes.

With the above interpretation, we therefore, conclude that any path flow $\tilde{f}_{P_W}^W\left(t^T\right)$ vector which solves system (7.68), is a multi-period user equilibrium (MPUE) flow. The proof follows from the *KKT* analysis and argument given above.

Furthermore, following the same lines of argument that led to Eq.(7.85) for the system problem, we arrive at the following result for the user problem:

$$\boxed{\sum_{p \epsilon P_w}\left(\tilde{\eta}_p^w\left(t^i\right)\right)\tilde{f}_p^w\left(t^i\right) = \delta^w\left(t^i\right)\hat{d^w}\left(t^i\right) \qquad \forall w \in W} \tag{7.91}$$

for any t^i. Eq.(7.91) is the network cost balance equation. Hence we summarize as follows: for any given t^i, the following condition holds for the user problem:

$$\boxed{\begin{array}{ll} \tilde{\eta}_p^w\left(t^i\right) + c_{t^j t^i}^w \geq \delta^w\left(t^j\right) & \forall w \in W, j \\ \sum_{p \epsilon P_w}\left(\tilde{\eta}_p^w\left(t^i\right)\right)\tilde{f}_p^w\left(t^i\right) = \delta^w\left(t^i\right)\hat{d^w}\left(t^i\right) & \forall w \in W \end{array}} \tag{7.92}$$

The first-best multi-period route-based pricing scheme

Here we use the term first-best to mean a network scenario where all routes are allowed to be tolled at all times. In such a setting, the system optimum is usually guaranteed given the nature the problem.

Now compare SP conditions (7.85) with UP conditions (7.92) and observe that the difference between them is the quantity

$\left(\bar{f}_p^w(t^i)\frac{d}{df_p^w(t^i)}\left(\bar{\eta}_p^w(t^i)\right)\right)$ seen in the analysis of the SP. Therefore, by adding the term

$$\left(f_p^w\left(t^i\right)\frac{d}{df_p^w\left(t^i\right)}\left(\eta_p^w\left(t^i\right)\right)\right)\Bigg|_{f_p^w(t^i)=\bar{f}_p^w(t^i)}$$

to the path travel cost $\tilde{\eta}_p^w(t^i) \quad \forall p \in P_w, w \in W$, the first-order optimality conditions of the user problem will exactly be the same as those of the system problem. This means that any flow pattern that solves the system problem will also solve the user problem $\left(i.e.\ \bar{f}_p^w(t^i) = \tilde{f}_p^w(t^i) \quad \forall p \in P_w,\ w \in W\right)$.

If we denote by $\theta_p^w(t^i)$ the optimal toll to be paid on route $p \in P_w$ when departing origin r during time t^i towards destination s, then the first-best optimal route toll can be given by

$$\bar{\theta}_p^w\left(t^i\right) = \left(f_p^w\left(t^i\right) \frac{d}{df_p^w\left(t^i\right)}\left(\eta_p^w\left(t^i\right)\right)\right)\Bigg|_{f_p^w(t^i)=\bar{f}_p^w(t^i)} \tag{7.93}$$

where $\bar{f}_p^w(t^i)$ is the solution of the SP.

Interpretation: The toll $\bar{\theta}_p^w(t^i)$ is the additional travel cost imposed on all the existing users of route $p \in P_w$ by an additional user on route $p \in P_w$, all departing from the same origin at the same departure interval t^i, and heading towards the same destination. Further observe from Eq.(7.82) that this quantity in (7.93) depends travel time cost and transfer cost of other departure times $t^j s$. This means that $\bar{\theta}_p^w(t^i)$ is not only the additional travel cost imposed on all the existing users of route $p \in P_w$ by an additional user on route $p \in P_w$ departing at t^i, but also additional travel cost imposed all users who depart at different departure time slot $t^j, j \neq i$. Note that, if this additional user does not depart during t^i, maybe another user who departs during t^j, $j \neq i$, may prefer to depart during t^i.

Therefore, by adding the toll $\bar{\theta}_p^w(t^i)$ to the cost of travel on route $p \in P_w$ for users departing during t^i, we now ensure that all users, before embarking on a trip, take into account the cost they incur and impose on other travellers by departing at the chosen time t^i. It turns out that $\bar{\theta}_p^w(t^i)$ as given in Eq.(7.93) is not the only possible toll that can achieve the system optimal flow $\bar{f}_p^w(t^i)$, in fact, there is an infinite number of toll vectors that can achieve this optimal flow, thus we state the following:

Corollary 2 *Suppose $\bar{f}_p^w(t^i)$ solves system (7.70), then for all departure times t^i, any route toll $\theta_p^w(t^i)$, $p \in P_w$ satisfying the following linear conditions will also induce the optimal route flow pattern $\bar{f}_p^w(t^i)$ as a multi-period user equilibrium (MPUE) flow pattern:*

$$\boxed{\begin{array}{ll} \left(\bar{\eta}_p^w\left(t^i\right) + c_{t^j t^i}^w + \theta_p^w\left(t^i\right)\right) \geq \delta^w\left(t^j\right) & \forall w \in W, j \\ \sum\limits_{p \epsilon P_w}\left(\bar{\eta}_p^w\left(t^i\right) + \theta_p^w\left(t^i\right)\right)\bar{f}_p^w\left(t^i\right) = \delta^w\left(t^i\right)\bar{d}^w\left(t^i\right) & \forall w \in W \end{array}} \tag{7.94}$$

where $\delta^w(t^j)$ is a free variable, and $\bar{d}^w(t^i)$ is the optimal demand of users departing origin r toward destination s during time t^i.

Proof: The proof simply follows from the *KKT conditions* of the SP and UP, and the argument given earlier in this Appendix. ■

Note that with Eq.(7.94), one can easily define secondary objectives on the path tolls, for example, fixing the total toll collected, minimizing the maximum route toll over all routes, etcetera.

For the OD-based pricing scheme, one only needs to replace the route-based tolls $\theta_p^w\left(t^i\right)$ in Eq.(7.94) with the OD-based tolls $\theta^w\left(t^i\right)$.

Chapter 8

Policy implications and discussions

People have realised that in the coming years, if something is not done, jammed traffic, pollution, noise, safety issues and other traffic externalities will only grow worse, with dire consequences. Due to financial, geographical, and political limitations, and the fact that even the expansion of the existing infrastructure may not lead to efficient use of transportation networks, it is envisaged that road pricing seems a viable option for achieving a more efficient use of the existing infrastructure. With all its potentials, road pricing has not gained all the supports it needed though. This lack of support is mainly due to how the pricing scheme is developed and perceived.

Our main motivation for the research carried out in this thesis stems from the fact that road pricing has long been modelled as a Stackelberg game, where the government or the toll operator decides on the pricing scheme leaving the road users and stakeholders affected by the scheme with little or nothing to do. Further, many road pricing schemes consider only single or two objectives without realising the effects of that on other traffic externalities. Moreover, many schemes have neglected the reactions and positions of some groups, actors or stakeholders if you want, and even the effect on businesses (private companies), during the development and implementation of the road pricing schemes. "Businesses", for example, "often do not have knowledge about how the policy might affect them, given the lack of reliable data. However, businesses represent a powerful interest group that opposes road pricing." In addition, environmental groups which had not historically made transportation a central focus of their effort is now joining the campaign (in approval or disapproval) of road pricing. As we have seen, public (and/or stakeholders) acceptance is widely recognised as a major barrier to widespread adoption of road pricing in most cities. Studies showed that road pricing proposals need to be perceived as benefiting drivers individually and not simply society at large.

In this thesis, we take into account that various stakeholders with different interest may be involved during the toll decision making or "debate". We also consider the interest of the road users since their involvement may change how motorists view the effect of pricing on them personally. This strategy enables us to build support for the road pricing scheme among elected officials, key stakeholders, and the general public. Taking the objectives of these various stakeholders and most important traffic externalities into our models, we hope that the road pricing schemes developed in this thesis will lead to a scheme that is fair and acceptable by the society. In fact, our study shows that a road pricing scheme that considers only one or part of the whole set of the traffic externalities, may do so at the detriment of other externalities. In particular, a road scheme that focusses in mitigating congestion, may lead to frequent road accidents and high traffic emissions. This in turn will lead to rejection of such proposal by, say, the environmental groups or activists.

Take away #1

The underlying models for a good road pricing scheme should take into account the (usually) conflicting interests of various stakeholders and the road users, and all traffic externalities. Further, the effect of the pricing scheme on businesses has to be well understood.

The fact that various stakeholders have different (and often contradicting) objectives, makes the problem a multi-objective problem (MOP), and the fact that we are dealing with more than one stakeholder, makes it a multi-actor problem. MOPs have been studied in traffic networks, and the most plausible or intuitive is to generate a set of the so called Pareto points to the MOP and present them to the policy or decision makers to choose a comprise (or desired) point. This is synonymous to the weighted sum method of solving MOPs, and it is always a question of what weights to choose for which objective. In the problem studied in this thesis, the stakeholders have different (and usually conflicting) interests. Our study shows that the stakeholders choice of (Pareto) points may be very much different from one another. This implies that a choice of a single "Pareto" point may not be accepted by some stakeholders.

This thesis took a different and novel direction to deal with the multi-stakeholder problem. We mimic a practical and political arena, where various stakeholders, each with his own objective, debate on the effect of the road pricing on the externalities in questions, on different user classes, on the environment, and society, on nearby traffic and networks, and so on. It is always during such discussion that the proposed pricing scheme is adopted, amended or dropped. We model the problem as a game allowing all the influential actors to participate in the game. As legislative powers vary from one country/government to another, it depends of course whether the implementation of road pricing needs legislative approval or not.

Take away #2

The traditional way of solving multi-objective problems, namely, choosing a point from a set of Pareto points is not suitable for developing a just and acceptable road pricing scheme.

Since a point in the Pareto set may not be accepted by all the stakeholders, the first set of questions is: (1) can we find a feasible toll point (not necessarily a Pareto point) that is acceptable by all the stakeholders, after all, only one toll pattern (or a solution point) has to be implemented, and (2) if such point exists, where is it in the toll solution space.

As stated before, the idea used in this study to answer these questions is to look at these stakeholders as players, where turn by turn, each player proposes a toll he thinks is optimal for his particular objective giving other stakeholders proposed tolls. The essence of this game is to look for a situation where after some play turns, the stakeholders can no longer improve their objectives by changing their current toll strategy. At this point (a Nash equilibrium point), the resulting cumulative toll pattern is a stable pattern (or an ideal point) since no actor can improve his objective by unilaterally changing his toll strategy. In other words, a toll pattern is stable if all the stakeholders are contented with it. Our study showed that such an ideal point does not exist in general. On the other hand, if suitable restrictions (such as conditions on the number of toll roads, on user classes, on the time of the day, on the level of tolls, etcetera) are placed on the tolling game, such an ideal point may exist. Note that this point may not be a Pareto point, meaning that the actions of uncoordinated players may lead to a sub-optimal (or worse) traffic situation. The actors as used here can as well play the role of autonomous cities and communities.

The implication of this is that in general (or in practice), we do not expect the "rational" stakeholders or autonomous cities to arrive at or agree on a toll pattern without some sort of cooperation or coalition in designing the scheme. If we assume that stakeholders suggest tolls in turns during a tolling scheme debate, it simply means that there will be an "endless" toll debate among them. Synonymously, cities connected by road networks and autonomous in their road pricing schemes will "endlessly" change their toll patterns in order to optimize their individual objectives.

Take away #3

- *Toll debate among stakeholders may result in a rat race or a tolling pattern that is acceptable by all the stakeholders, but then, may worsen the existing traffic situation.*
- *Further, decentralized tolling schemes by cities may result in a rat race, or a scheme that may worsen the traffic situations in these cities, thereby defeating the aim of road pricing.*

Since the stakeholders or the actors do not agree on a tolling scheme in general, and even if there is a tolling pattern accepted by all the stakeholders, this ideal pattern may then be far from the system (Pareto) optimal solutions. We developed a novel mechanism that will induce a toll pattern that is system optimal and stable among the stakeholders. The mechanism we developed ensures that a given toll pattern is system optimal, and at the same time stakeholder optimal for all stakeholders. This makes this toll pattern a stable one and acceptable by all the actors involved. You may recall that these stakeholders may have conflicting objectives, and how can a single toll pattern be optimal for all stakeholders? The answer lies on the fact that the mechanism aligns the objectives of all the stakeholders to that of the system or Grand leader's objective using a taxing mechanism just as in a Stackelberg game, where the leader aligns the followers' actions to his objective using, for example, tolls as in the case of road pricing. The fact that we are dealing with stakeholders at different levels, makes the problem a multi-level problem.

This implies that a "central government" for example, can influence and check the actions of non-cooperative actors, aligning their objectives (without asking them to cooperate) to achieve a certain and acceptable aim. The Grand leader or the "central government" can either tax or give subsidies to the stakeholders in order to achieve his (and a common) aim for the stakeholders using the mechanism.

The U.S. Department of Transportation's Urban Partnership Program sets aside about $1 billion from a dozen highway and transit programs for applicants that satisfied criteria related to the "four 'T's" of Tolling, Transit, Telecommuting, and Technology. This subsidy is of course to lure cities and states into a federal government program.

Take away #4

- *It is possible to centralize a decentralized tolling scheme using a mechanism design, achieving a scheme that is stable, acceptable and "optimal".*
- *The use of subsidies to steer stakeholders' actions could lead to a corrupt system, where one or some of the stakeholders would lobby the Grand leader or the "central government" to use the taxing mechanism in their favour.*

Link-based and route-based tolling schemes have their shortcomings. People have always questioned and criticized some of the underlying operating principles of these schemes. For example

- what happens when a temporal road disturbance such as accidents, road constructions and repairs occur, and people may have to change their usual travel pattern or route? This often leads to extra travel cost for the users.
- what happens when a road segment leading to residential street is a tolled road, and residents always have to pay each time to go out or come in?
- can we find a tolling scheme that avoids the huge investment costs of road pricing road infrastructures?
- can we find a scheme that encourages intra-mode transfer, and would increase the mode share of public transport?

In search for an answer, we developed a novel road pricing scheme that charges a road user based on his origin and destination. The origin destination-based scheme optimally regulates traffic in and out of a transportation infrastructure according to the time of the day enabling a peak-hour spread. The proposed scheme has one downside though; it does not (in general) optimize the route split among users.

Take away #5

An OD-based road pricing scheme presents a promising and acceptable model for the new generation road pricing schemes.

How to use the fund generated from road pricing programs has always been a point of debate, and this may be a point of disagreement among the stakeholders, and consequently may lead to dropping of the pricing scheme. It has always been said that the revenue generated will be invested back into the transportation system so as not to increase the social cost for the drivers. It is argued that such investment will benefit the drivers, but then, study shows that drivers are unlikely to feel that the value of congestion reduction is worth the fee. Their view is by no means irrational since pricing usually "make(s) travellers worse-off before the usage of revenues is accounted for." Moreover, investing the revenue on road developments still needs to demonstrate why a select group of drivers should pay while others do not.

Further, stakeholders have questioned whether the revenues generated will be distributed effectively so as to justify equity among the regions, the users and the pricing zones. It is mostly for this reason that fuel or gasoline taxes presents a sound argument as a "fair" means to generate funds for road maintenance. On the other hand, a distance-based pricing scheme will ensure that the more one uses the road, the more he pays, and such a scheme may be seen to be a fair and equitable system of generating revenue for road maintenances. It should be noted that when road pricing is in place, some of the broad-based taxes such as sales taxes, has to be stopped since sales taxes, for example, are clearly less equitable. Sales taxes in particular penalizes non-users. Creating public awareness of the benefit of a road pricing scheme is one way to increase public acceptance of the scheme. Note that the "efficiency and equity of any proposed road charge depend on the travel market for which the charge is proposed. Moreover, what is acceptable in Europe may not be in North America or vice versa." Further, road pricing schemes should map out from the onset how to charge foreign vehicles.

Take away #6

- *A good proposal for a road pricing scheme should contain a fair or equitable use of the revenue generated. The scheme itself should also be viewed as fair or equitable by stakeholders.*
- "*In order to achieve process equity, transparency in the decision-making process, in addition to allowing input from all potentially affected individuals or groups representing them, is required.*"

Chapter 9

Conclusions and recommendations

We will now revisit the research questions of Chapter 1 and see how far we have done justice to those questions.

Research questions revisited

We will give pointwise answers to the research questions.

1. Under static traffic assignment (STA), the thesis addresses the following questions:
    - What happens when stakeholders do not cooperate in toll setting?
        - If the stakeholders do not cooperate, it is likely that they do not come to a compromise toll pattern in general. On the other hand, if they do come to a compromise toll pattern, this toll pattern may lead to a network situation that is far from the optimal flow (Chapter 4).
    - Under which conditions can the existence of a Nash equilibrium (NE) be guaranteed?
        - We could not establish concrete and sufficient conditions to ensure the existence of NE mainly because the players' objectives are in general not convex in the toll strategies. This condition is enough to "ruin" the existence of NE. However, under suitable conditions and restrictions the tolls, NE may exist (Chapter 4).
    - Can we design a mechanism that induces a Nash equilibrium between the actors?
        - Yes, we designed a taxing mechanism that induces a NE among the actors. We created the so called *Grand leader* (*GL*) that oversees the affairs of the actors. Using taxes, the GL could induce the NE among the actors in the same way the actors induce Wardrop's equilibrium among the road users with the aid of road tolls (Chapter 5).
    - When and how can a cooperative solution concept in the form of a common road pricing scheme be found?
        - We can expect a cooperative solution under the inducing mechanism described in Chapter 5. That is, when the *GL*, with the aid of the taxes, indirectly persuade actors to a cooperative (Chapter 5).
    - If the stakeholders agree to cooperate, how would they share the benefits?

  - The answer to this question lies solely on the stakeholders, and most importantly, the objectives of these stakeholders. We found that a stakeholder is likely to accept an offer (say, from a coalition), only when the offer is better than what he can obtain standing alone in the road pricing game (Chapter 4).
- Can we design a mechanism that induces a cooperative outcome on otherwise non-cooperative actors, and thus achieve the system optimum or any other prescribed state within the system?
  - Yes, in Chapter 5, we designed a mechanism that induces a NE among the actors. The mechanism also ensures that the NE point is a system optimal point, or some other point desired by the *Grand leader* (Chapter 5).
- Which coalitions among the stakeholders are likely to be formed in a cooperative concept?
  - A coalition is likely to be formed if such coalition will prove to be a stable coalition. A coalition is stable if there is a "wealth" allocation scheme among the players of this coalition such that they are all better off staying in this particular coalition than in any other coalition (Chapter 4).
- What can we say about the various classical solution concepts from cooperative game theory, such as core, nucleolus and bargaining sets?
  - Core and bargaining sets of the road pricing game are discussed in Chapter 4. The game may have a core under the conditions described in Chapter 4 of this thesis (Chapter 4).

2. Equity issues:

   Can we design a tolling scheme such that:
   - People do not pay "unnecessarily high" tolls because of where they live or work?
     - Yes, and OD-based toll has the potentials of take care of the problem (Chapter 7).
   - Flat tolls or user-specific tolls: which is more acceptable and to whom?
     - Was not investigated in this thesis, recommended for future research (Next section)
   - OD-based tolls or link-based tolls: which is better from both the system's and users' perspectives?
     - OD-based tolling scheme has some obvious benefits over the link-based counterpart, but the research on how it will be accepted by the stakeholders and the road users is still lacking, thus a recommendation for future research (Next section)
   - Finally, can we find a tolling scheme that leaves every player (including the road users) contented?
     - Indeed, the tolling scheme under the optimal Nash inducing mechanism can create a "satisfying" outcome. Further, allocation rules leading to stable coalitions may also lead to contented solutions (Chapter 4 & 5).

3. Implementation and practical application of our model
    - How does the model apply to a realistic network.
        - The models developed in this thesis were all tested on a small-size example networks. The models on OD-based road pricing was tested on the well-known Nguyen and Dupuis network. The application on a real life network is recommended for future research (Next section).
4. Model extension
    - In which other domains might our models be applicable?
        - The model developed in this thesis, especially the models of Chapter 5 have wide applications. For example, the optimal inducing mechanism can also be used to induce a system optimal performance in the following scenarios:
            * Malicious nodes in car to car communication where cars exchange data/information within a limited time frame (Schwartz et al. [58, 56, 55, 57]).
            * Local authorities tolling separate regions of the network.
            * Energy producers in the energy market liberalization problem.
            * Agents in the principal-agent model.
            * Internet providers in the providers-subscribers Internet price setting problem.
            * Competition of firms over the same market shares.
            * Employees that have flexibility on the number of workdays.
5. What are the policy implications of the study?
    - What can the government, stakeholders, and road users learn from it?
        - The policy implications of the study are given in Chapter 8 of this thesis (Chapter 8)
    - How feasible are the models?
        - The models developed in this thesis look feasible and promising Further, the models/study have the potentials of alleviating some issues that have hindered the adoption and approval of road pricing schemes in most cities.

Future research directions

With the road pricing models ready, the next line of research will be to implement it in a real life test case to see the impact of the models developed in this thesis. Testing such models will be easy in cities where road pricing schemes are up and running.

In Chapter 7, we analyse the OD-based pricing scheme under multi-period traffic assignment model. It will be nice if the models of this thesis are tested in a fully dynamic environment though.

Further, the models developed in this thesis revolve around classical optimization, heuristic counterparts may be needed for huge network applications.

In Chapter 8, we noted the policy implications of the study carried in this thesis, but then, it will be good if research is carried out to investigate the reactions of the government, the stakeholders and the users on the type of pricing mechanisms described in this thesis.

Bibliography

[1] Armstrong-Wright, A., 1986. Road pricing and user restraint: opportunities and constraints in developing countries. Transportation Research Part A: General 20 (2), 123–127.

[2] Arnott, R., Palma, A. D., Lindsey, R., 1993. A structural model of peak-period congestion: A traffic bottleneck with elastic demand. The American Economic Review 83 (1), 161–179.

[3] Aumann, R. J., 1959. Acceptable points in general cooperative n-person games. Vol. 4. In: Tucker, A.W., Luce, R.D. (eds) Contributions to the Theory of Games IV. Princeton: Princeton University Press.

[4] AVV, 1998. 1998, "Opinion on travel time valuations of persons", Rotterdam. Ministry of Transport and Water Management and Ministry of Economic Affairs, 2004. "Supplement guide EIA-direct effects", Den Haag.

[5] AVV, 2003. Prestaties Nederlandse wegennet. De ontwikkeling van het wegverkeer, de wegcapaciteit en congestie in verleden en toekomst. Tech. rep., Rotterdam, Netherlands.

[6] AVV, 2004. Fileverkenning. De ontwikkeling van vertragingen op het Nederlandse autosnelwegennet. Tech. rep., Rotterdam, Netherlands.

[7] Beckmann, M., McGuire, C., Winsten, C., 1955. Studies in the Economics of Transportation. Yale University Press, New Haven. CT, New Haven. CT.

[8] Bergendorff, P., Hearn, D., 1997. Congestion toll pricing of traffic networks. Lecture Notes in Economics and Mathematical Systems, 51–71.

[9] Berkum, E. V., Vlist, M. V. D., 2003. A Framework to Determine the Possibilities for Slot Allocation in Car Travel. In: 19th Dresden Conference on Traffic and Transportation Science. Dresden, the Netherlands, pp. 1–11.

[10] BOVAG, 2007. Mobiliteit in cijfers 2006. Tech. rep., Netherlands.

[11] Boyce, D., Lee, D., Ran, B., 2001. Analytical models of the dynamic traffic assignment problem. Networks and Spatial Economics, 377–390.

[12] Braess, D., Nagurney, A., Wakolbinger, T., Nov. 2005. On a Paradox of Traffic Planning. Transportation Science 39 (4), 446–450.

[13] Cheng, F., Li, D., 1996. Genetic algorithm and game theory for multiobjective optimization of seismic structures with/without control. In: Proc., 11th World Conf. on Earthquake Engineering,. p. 1503.

[14] Coello, C. A. C., 2006. Carlos A. Coello Coello. IEEE Computational Intelligence Magazine (February 2006), 28–36.

[15] Deb, K., 2001. Multi-objective Optimization Using Evolutionary Algorithms (Wiley Interscience Series in Systems and Optimization). Wiley-Blackwell.

[16] Deb, K., Pratap, A., Agarwal, S., 2002. A fast and elitist multiobjective genetic algorithm: NSGA-II. IEEE Transactions on Evolutionary Computations 6 (2), 182–197.

[17] den Boer, L., Schroten, A., 2007. Traffic noise reduction in Europe Health effects , social costs and technical and policy options to reduce road and rail traffic noise. Tech. Rep. August, CE Delft, Solution for environment, economy and technology, Delft, The Netherlands.

[18] Dimitriou, L., Tsekeris, T., 2007. Dynamic congestion pricing based on user-optimal stochastic learning models. In: Kuhmo Nectar Conference. No. July. pp. 1–11.

[19] Drissi-Kaitouni, O., 1993. A variational inequality formulation of the dynamic traffic assignment problem. European journal of operational research 71, 188–204.

[20] Ecorys, 2006. Netwerkanalyse Stedendriehoek. Verkenning voor de periode 2010- 2020. Hoofdrapport. Tech. rep., Rotterdam, Netherlands.

[21] Ferrari, P., Feb. 1999. A model of urban transport management. Transportation Research Part B: Methodological 33 (1), 43–61.

[22] Hau, T. D., 1992. Economic Fundamentals of Road Pricing.

[23] Hearn, D. W., Ramana, M. V., 1998. SOLVING CONGESTION TOLL PRICING MODELS. Equilibrium and Advanced Transportation Modeling, Kluwer Academic, 109–124.

[24] Hu, X., Ralph, D., Sep. 2007. Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices. Operations Research 55 (5), 809–827.

[25] InfoMil, 2007. Handleiding webbased CAR: Versie 7.0. CAR II (Calculation of Air pollution Road traffic) model version 7.0 (2009 data). See http://car.infomil.nl. Tech. rep.

[26] Ismail, I., El_Ramly, N., El_Kafrawy, M., 2007. Game Theory Using Genetic Algorithms. In: Proceedings of the World Congress on Engineering. Vol. I. pp. 7–10.

[27] Joksimovic, D., 2007. Dynamic bi-level optimal toll design approach for dynamic traffic networks. T2007/8, September 2007, TRAIL Thesis Series, The Netherlands.

[28] Joksimovic, D., Bliemer, M., 2005. Dynamic road pricing optimization with heterogeneous users. ERSA conference papers.

[29] Jonasson, H., 2003. Source modeling of road vehicles. Harmonoise work package 1.1.

[30] Koh, A., 2011. Universities of Leeds , Sheffield and York An Evolutionary Algorithm based on Nash Dominance for Equilibrium Problems with Equilibrium Constraints. Applied Soft Computing.

[31] Krawczyk, J., Zuccollo, J., Dec. 2006. NIRA-3: An improved MATLAB package for finding Nash equilibria in infinite games.

[32] Lawphongpanich, S., Hearn, D. W., Jul. 2004. An MPEC approach to second-best toll pricing. Mathematical Programming 101 (1), 1–23.

[33] Leyffer, S., Munson, T., Aug. 2010. Solving multi-leader-common-follower games. Optimization Methods and Software 25 (4), 601–623.

[34] Liu, G. P., Yang, J.-B., Whidborne, J. F., 2003. Multiobjective Optimization and Control. Research studies press Ltd, Hertfordshire, England.

[35] Luk, J., Chung, E., 1997. Public acceptance and technologies for road pricing. Tech. rep., ARR 307, ARRB Transport Research.

[36] Mardle, S., Miettinen, K. M., Feb. 2000. Nonlinear Multiobjective Optimization. Vol. 51. Kluwer Academic Publishers, Massachusetts.

[37] May, A., Shepherd, S., 2002. Optimal locations for road pricing cordons. Road Transport Information (486), 69–73.

[38] Milchtaich, I., Nov. 2006. Network topology and the efficiency of equilibrium. Games and Economic Behavior 57 (2), 321–346.

[39] Morrison, S., 1986. A survey of road pricing. Transportation Research Part A: General 20 (2), 87–97.

[40] Nash, J., 1951. Non-Cooperative Games. Annals of Mathematics 54 (2), 286–295.

[41] Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V. V., Jul. 2007. Algorithmic game theory. Cambridge University Press, New York, USA.

[42] Ohazulike, A. E., 2009. Multi-Objective Road Pricing Problem: A Cooperative and Competitive Bilevel Optimization Approach. Master Thesis, University of Twente.

[43] Ohazulike, A. E., Bliemer, M. C. J., Still, G., Berkum, E. C. V., 2012. Multi-Objective Road Pricing : A Game Theoretic and Multi-Stakeholder Approach. In: 91st annual meeting of the Transportation Research Board (TRB) Conference, Washington D.C. pp. 12–0719.

[44] Ohazulike, A. E., Still, G., Kern, W., Van Berkum, E. C., 2012. Multi-stakeholder road pricing game: solution concepts. International Journal of Computational and Mathematical Sciences . World Academy of Science Engineering and Technology 6, 1–12.

[45] Ohazulike, A. E., Still, G., Kern, W., van Berkum, E. C., 2013. An origin-destination based road pricing model for static and multi-period traffic assignment problems. Transportation Research Part E: Logistics and Transportation Review 58, 1–27.

[46] Olsder, G. J., May 2009. Phenomena in Inverse Stackelberg Games, Part 1: Static Problems. Journal of Optimization Theory and Applications 143 (3), 589–600.

[47] Olsder, G. J., May 2009. Phenomena in Inverse Stackelberg Games, Part 2: Dynamic Problems. Journal of Optimization Theory and Applications 143 (3), 601–618.

[48] Periaux, J., Chen, H., Mantel, B., Sefrioui, M., Sui, H., May 2001. Combining game theory and genetic algorithms with application to DDM-nozzle optimization problems. Finite Elements in Analysis and Design 37 (5), 417–429.

[49] Public Transport Users Association Inc. (PTUA), 2010. Common Urban Myths About Transport. Available at: http://www.ptua.org.au/myths/zones.shtml [Accessed January 2014].

[50] Ran, B., Hall, R., Feb. 1996. A link-based variational inequality model for dynamic departure time/route choice. Transportation Research Part B: 30 (1), 31–46.

[51] Regeling van de Minister van Volkshuisvesting, R. O. e. M., 2006. Bijlage III Behorende bij hoofdstuk 3 Weg van het Reken- en meetvoorschrift geluidhinder 2006, 1–58.

[52] Road-Design-Factor, 2009. Available From: http://www.asphaltwa.com/wapa_web/modules/04_design_factors/04_loads.htm#esal [Accessed August 2009].

[53] Saguan, M., Plumel, S., Dessante, P., Glachant, J., Bastard, P., 2004. Genetic algorithm associated to game theory in congestion management. In: 8th International Conference on Probabilistic Methods Applied to Power System. pp. 415–420.

[54] Schaller, B., 2010. New York City's Congestion Pricing Experience and Implications for Road Pricing Acceptance in the United States. Transport Policy 17, 266–273.

[55] Schwartz, R., Ohazulike, A., Sommer, C., Scholten, H., Dressler, F., Havinga, P., 2012. Fair and Adaptive Data Dissemination for Traffic Information Systems. In: Fourth IEEE Vehicular Networking Conference 2012 (IEEE VNC 2012). IEEE Intelligent Transportation Systems Society. ISBN 978-1-4673-4996-3., Seoul, South Korea, pp. 1–8.

[56] Schwartz, R. S., Ohazulike, A. E., Scholten, H., May 2012. Achieving Data Utility Fairness in Periodic Dissemination for VANETs. In: IEEE 75th Vehicular Technology Conference (VTC2012-Spring). IEEE Vehicular Technology Society. ISSN 1550-2252 ISBN 978-1-4673-0989-9, Yokohama, Japan, pp. 1–5.

[57] Schwartz, R. S., Ohazulike, A. E., Sommer, C., Scholten, H., Dressler, F., Havinga, P., 2014. On the applicability of fair and adaptive data dissemination in traffic information systems. Ad Hoc Networks 13, 428–443.

[58] Schwartz, R. S., Ohazulike, A. E., van Dijk, H. W., Scholten, H., Sep. 2011. Analysis of Utility-Based Data Dissemination Approaches in VANETs. In: 4th International Symposium on Wireless Vehicular Communications (WIVEC 2011) - VTC 2011 Fall. IEEE Vehicular Technology Society. ISSN 1090-3038 ISBN 978-1-4244-8328-0, San Francisco, CA, USA, pp. 1–5.

[59] Sefrioui, M., 2000. Nash genetic algorithms: Examples and applications. Proceedings of the 2000 Congress on Evolutionary Computation 1, 509 – 516.

[60] Sharma, S., Ukkusuri, S. V., Mathew, T. V., 2009. Pareto Optimal Multiobjective Optimization for Robust Transportation Network Design Problem. Transportation Research Record: Journal of the Transportation Research Board (2090), 95 – 104.

[61] Small, K., 1982. The scheduling of consumer activities: work trips. The American Economic Review 72 (3), 467–479.

[62] Stankova, K., 2009. On Stackelberg and Inverse Stackelberg Games: & Their Applications in the Optimal Toll Design Problem, the Energy Markets Liberalization Problem, and. Ph.D. thesis.

[63] Staňková, K., Bliemer, M., Olsder, G. J., 2006. Inverse Stackelberg Games and Their Application to Bilevel Optimal Toll Design Problem. 12th International Symposium on Dynamic Games and Applications. Sophia, 1–2.

[64] Steuer, R., 1986. Multiple Criteria Optimization: Theory, Computation, and Application. John Wiley & Sons, New York, New York, USA, pp. 158–220.

[65] Tan, K.-K., Yu, J., Yuan, X.-Z., Sep. 1995. Existence theorems of nash equilibria for non-cooperative n-person games. International Journal of Game Theory 24 (3), 217–222.

[66] Verhoef, E., 2000. Second-best congestion pricing in general networks. Tech. rep., Tinbergen Institute, Netherlands.

[67] Verhoef, E., Bliemer, M., Steg, L., 2008. Pricing in Road Transport: A Multi-disciplinary Perspective. Edward Elgar Publishing Ltd.

[68] Verhoef, E., Small, K., 2004. Product differentiation on roads: constrained congestion pricing with heterogeneous users. Journal of Transport Economics and Policy 38 (1), 127–156.

[69] Verhoef, E. T., May 2002. Second-best congestion pricing in general static transportation networks with elastic demands. Regional Science and Urban Economics 32 (3), 281–310.

[70] Vermeulen, J. P., Boon, B. H., Essen, H. P., Boer, E. L., Dings, J. M., Bruinsma, F. R., Koetse, M. J., 2004. Aanpassing prijspeil door ECORYS (price adjustment by ECORYS): De prijs van een reis (The price of a trip).

[71] Warburton, A. R., Aug. 1983. Quasiconcave vector maximization: Connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives. Journal of Optimization Theory and Applications 40 (4), 537–557.

[72] Wardrop, J., 1952. Some theoretical aspect of road traffic research. Proceedings of the Institute of Civil Engineers 1, 325–378.

[73] While, L., Hingston, P., 2006. A faster algorithm for calculating hypervolume. IEEE Transactions on 10 (1), 29–38.

[74] Wisman, L., Eric van Berkum, 2008. Multi-objective optimisation of traffic systems Modelling external effects. TRAIL Research School Delft (October).

[75] Wismans, L. J. J., Van Berkum, E. C., Bliemer, M. C. J., 2011. Comparison of Multiobjective Evolutionary Algorithms for Optimization of Externalities by Using Dynamic Traffic Management Measures. Transportation Research Record: Journal of the Transportation Research Board (2263), 163 – 173.

[76] Yan, H., Sep. 1996. Optimal road tolls under conditions of queueing and congestion. Transportation Research Part A: Policy and Practice 30 (5), 319–332.

[77] Yang, H., Huang, H. J., 2005. Mathematical and Economic Theory of Road Pricing. Elsevier, UK.

[78] Yildirim, M. B., 2001. CONGESTION TOLL PRICING MODELS AND METHODS FOR VARIABLE. Dissertation, University of Florida, USA.

[79] Yildirim, M. B., Hearn, D. W., Sep. 2005. A first best toll pricing framework for variable demand traffic assignment problems. Transportation Research Part B: Methodological 39 (8), 659–678.

Summary

Over the past years, vehicle ownership has increased tremendously leading to increase in traffic externalities such as congestion, emission, noise, safety, etcetera. By 2050, it is expected that more than 9 billion people will be living on Earth, up from 7 billion today. Asia's fast growing cities will absorb much of this growth with three out of four people living in urban centres. Billions of people will live above poverty lines, and will be able to afford luxuries. Development will reach to many places on Earth, demand for a better life will rise, car/vehicle ownership will increase leading to high demand for road capacities and infrastructures, yet supply for these road capacities and infrastructures is not going to increase in the same rate as their demand. Further, this increase in vehicle ownership will escalate the traffic externalities. In fact, due to geographical, financial and political reasons, most of the infrastructures will remain unchanged even when the demand is out pacing supply for these infrastructures. On the other hand, shift to cleaner energy resources may reduce emissions, but still will not remove cars out of the roads, and therefore, still leaves a great part of the externalities in place.

The above mentioned reasons led economists to start thinking of how to mitigate these externalities without depending on the physical expansion of infrastructures. One of the first ideas was to levy parking charges to deter cars from entering into some (usually urban) zones. With the increase in development, and quest for better life, car ownership increased, and, in fact, the benefits of owning a car outweigh the parking charges, and the subsidies received by employees from their employers for transportation fares cushion the effect of parking charges. These reasons make it impossible to tackle congestion problems with parking charges. Furthermore, parking fees do not depend upon the traffic volume or distance travelled, neither do they depend on the environmental characteristics of the vehicle. Traffic in transit through a congested area is not affected by parking fees at all. These show that parking fees are not so effective in battling traffic externalities.

Failure of parking charges to mitigate congestion gave birth to what we today know as road pricing. Road pricing is a scheme that defines charges on segments of a given transportation network in order to efficiently route users into and throughout the network. The scheme determines which of the segments to charge, how much to charge, and finally, when to charge. Road pricing schemes have many success stories since the inception. They have helped cities recover from frustrating traffic situations, generate funds for financing transport infrastructures, and even mitigate some externalities that were not part of the initial motives.

These days, due to high car ownerships, many traffic externalities (such as noise, safety, emission, etcetera) earlier ignored when developing road pricing models,

now have serious economic and health concerns. It is therefore, vital to include these externalities in making a good road pricing scheme. Successful implementation of any road pricing scheme depends mostly on two factors, namely, political and user acceptability of the proposed scheme. We have seen in practice that parliamentary debate on which of the road pricing schemes to implement and which of the externality the scheme mitigates had often led to a "still birth" of road pricing implementation. In addition, road users mostly see road pricing as an extra tax burden, and have most times protested against it. The debate in the parliament arises since stakeholders in the house have different and usually conflicting interests about road pricing aims, otherwise there will be no debate. In particular, a stakeholder may favour a scheme that mitigates congestion, while the other favours a scheme that keeps the environment clean and green, and so on. A toll pattern that mitigates one traffic externality may escalate the other, so such debate is frustrating and often lead to a "no deal" end. To make matters worse, a toll pattern that efficiently distributes traffic in one "sovereign" state/region may worsen the traffic situation in another (nearby) state/region. Therefore, for a road pricing scheme to be acceptable, it has to carry along all the stakeholders and the states involved. This means that we need pricing patterns and/or schemes that deal with conflicting interests of the stakeholders, the regions and the users, and leave every participating actor contented.

In search for such pricing schemes, game theoretic approach presents promising models. In this thesis, we have developed road pricing models and mechanisms to solve problems arising due to the conflicting interest of actors in developing road pricing schemes. We first investigate why stakeholders usually do not come to a compromising tolling pattern, and the answer lies in the fact that the debate (among the policy makers) on what toll pattern to adopt has in general no Nash equilibrium (NE). In particular, given a toll pattern and hence a traffic situation, we can always find a stakeholder (or region) who will be better off choosing a different toll pattern from the set of unbounded tolls. Although boundedness of tolls may enforce NE, we found that it still can create a cyclic game among the stakeholders leading to a no-Nash equilibrium game. This means that actions of uncoordinated stakeholders or regions may create insatiable or non-optimal tolling schemes, leading to instability of such schemes. We developed a model that takes as input the network of interest, the objectives of the stakeholders and the road users, and the conditions of operations, and then presents to the stakeholders a stable tolling scheme/pattern that leaves each of them contented. The models developed in Chapter 3 only define stable state tolls for the stakeholders, but do not guarantee the existence of Nash equilibrium.

Since under normal conditions we cannot guarantee NE among the actors, and meanwhile NE if exists may not be Pareto optimal, we designed a mechanism that induces NE among these actors. Interestingly, the NE inducing mechanism further ensures that the induced NE toll pattern is "optimal" for all stakeholders and at the same time Pareto optimal for the global network. We call this mechanism optimal Nash inducing mechanism. So when the issue of road pricing arises, the mechanism designs a tolling scheme for the given network, and presents to stakeholders a tolling pattern that is a Nash equilibrium. In this way, stakeholders (or regions) cannot improve on the tolling pattern and thus do not have any sense of unfairness, and this, sometimes, may lead to some degree of satisfaction.

Above all, the inducing mechanism is Pareto or (if you like) system optimal. This means that with this new model and mechanism, we have circumvented the frustrating debates to arriving at (equilibrium or) a compromise tolling pattern, which we know of course may not exist. The fact that the induced NE toll pattern is "optimal" for each of the stakeholders and Pareto optimal for the global system makes the scheme more likely to be accepted by the stakeholders and regions. We have also included users' interests such as low road tolls, no tolls for some roads, and equity issues (like different values of time for different income classes) and so on, to ensure user acceptability of the mechanism.

The optimal Nash inducing mechanism can be mimicked in dealing with many real life problems, for example; inducing optimal performance in the following set-up:

1. In telecommunication networks where cars equipped with sensors exchange (say) traffic and environmental information within a limited time frame. 2. Local authorities tolling separate regions of the network. 3. Energy producers in the energy market liberalization problem. 4. Agents in the principal-agent model. 5. Internet providers in the providers-subscribers Internet price setting problem. 6. Competition of firms over the same market shares. 7. Employees that have flexibility on the number of workdays.

See Chapter 5 for a detailed explanation of how we can mimic the optimal Nash inducing mechanism in these instances.

It is interesting to know that the game theoretic model presented in this thesis has given birth to a new way of solving multi-objective problems (MOPs). As we mentioned in Chapter 6 of this thesis, it is always desirable to list all possible solutions of an MOP to enable decision makers to choose suitable points (usually on the Pareto front). Most existing algorithms that perform that task of listing all solution points depend on the principle of Pareto dominance. This principle is the basis of most genetic algorithms, which have been robust and powerful in solving MOPs. When the number of objectives increases, these algorithms find it complex to handle the Pareto dominance, and hence begin to deteriorate. In fact, as soon as the objectives exceed four in number, the algorithms start degenerating. Given an MOP, we represent each of the objectives as a player and employ the game theoretic approach described in this thesis. Though the Nash game model does not ensure the generation of all non-dominated solutions, the competition among the actors (where each actor searches for the best solution given what other actors are doing) tends to draw the solution points near to the Pareto front. In our test case, we found that all solutions generated during the game either lie on the Pareto front or in the neighbourhood of the Pareto front, asserting the consistency with the game approach. This implies that good solutions are generated at an early stage during the game which is rarely the case in genetic algorithms. Further, the game mechanism we describe does not deteriorate with the number of objectives, and has nothing to do with Pareto dominance. We thus conclude that the game theoretical approach presents a promising method for quick generation of (non-dominated) solutions for multi-objective problems. The next line of research is to incorporate the nice features of genetic algorithms to that of the game approach which we believe will give birth to a powerful tool for MOPs.

Sometimes temporal road disturbances such as accidents, road constructions and repairs occur, and people may have to change their usual travel pattern or route. This often leads to extra travel cost for the users. This may lead to complications for a system that has road pricing schemes. A link-based pricing scheme means that users may now need to pay more for travelling on a link they did avoid initially. Similarly, a route-based or kilometre-based pricing also put more charges on the users since they now may need to travel more kilometres to complete their trips. On its own, a link-based tolling scheme may leave a user with no option, for example, where he has to pay for driving into his street because a road that connects him to his street has a price tag on it, and he has no alternative road, or even pays more for using an alternative road. Studies show that these types of complications and feelings of unfairness have made road pricing schemes unpopular in many countries and cities, notwithstanding its enormous potentials. Further, the cost of implementing link-based or route-based pricing could, in fact, be huge, and talks on implementation of road pricing schemes have stalled in many cities due to the huge financial implications. In this thesis, we have developed a tolling scheme that does not depend on which link or route you use during a trip, but on your origin and destination. The scheme has little data in its memory, only noting your origin and destination and not tracking your travel trajectories. This feature alone eliminates some privacy issues and saves cost in terms of data storage/management. The scheme does not involve building of tollbooths or mounting any equipment on the road, instead it involves a small electronic (a kind of GPS) equipment on a car that notes the origin and destination of a trip (like the public transport card of The Netherlands). In addition, since the origin-destination (OD) tolling scheme does not depend on the link or route used, it then means that our earlier critics of link-based and route-based schemes are now addressed by this new developed scheme. This means users need not to pay more due to a temporal road disturbance or where his residence or workplace is located. The OD-based scheme optimally regulates traffic in and out of a transportation infrastructure according to the time of the day enabling a peak-hour spread. The proposed scheme has one downside though; it does not (in general) optimize the route split among users.

In conclusion, this thesis sheds light on various road pricing models, and has used game theory to model the “power tussle” among several stakeholders who naturally have interests in road pricing schemes. It has also included the interests of the road users in the main block of the toll decision algorithms, allowing for a toll pattern that are less likely to face public disapproval. As a basis for practical implementation, the thesis provides all the necessary mathematical models for the various road pricing schemes.

Samenvatting

Het autobezit is de laatste jaren enorm gegroeid, wat heeft geleid tot allerlei externe effecten, zoals files, uitstoot van schadelijke gassen, geluidsoverlast, verkeersonveiligheid, enzovoort. De wereldbevolking neemt naar verwachting toe van 7 miljoen mensen nu, naar 9 miljoen mensen in 2050. De snel groeiende steden in Azië zullen veel groei opnemen, waarbij driekwart van de mensen in steden leeft. Miljarden mensen zullen boven de armoedegrens leven, en zullen zich luxe kunnen veroorloven. Over de hele wereld zal men zich verder ontwikkelen, zal autobezit toenemen en daardoor zal er een grote vraag zijn naar capaciteit op de weg, terwijl de capaciteit van deze infrastructuur niet zo snel zal stijgen als de vraag ernaar. Daarbij zal de stijging in autobezit leiden tot verergering van de externe effecten van verkeer. De meeste infrastructuur zal niet wijzigen, vanwege geografische, financiële en politieke redenen, ook al stijgt de vraag harder dan het aanbod van infrastructuur. Het gebruik van schone energie zal weliswaar emissies verminderen, de auto's worden er niet door van de weg gehaald, waardoor een groot deel van de externe effecten blijft bestaan.

De genoemde redenen hebben er toe geleid dat economen zijn gaan denken hoe deze externe effecten te verminderen, zonder de infrastructuur fysiek uit te breiden. Invoering van betaald parkeren is een eerste aanpak geweest om auto's te ontmoedigen bepaalde gebieden binnen te rijden (vaak in binnensteden). Door toegenomen welvaart steeg het autobezit en de voordelen van het autogebruik wogen op tegen deze parkeertarieven. Daarbij komt dat werkgevers vaak parkeerkosten vergoeden voor werknemers, waardoor het effect van parkeerbeleid deels teniet wordt gedaan. Daardoor kunnen files niet volledig met parkeertarieven worden bestreden. Daar komt bij dat parkeertarieven nu niet afhangen van de verkeersintensiteit, gereisde afstand of milieukenmerken van het voertuig. Tenslotte wordt doorgaand verkeer in een betaald parkeren gebied niet door deze parkeertarieven beïnvloed. Dit laat zien dat parkeertarieven niet erg effectief zijn om externe effecten van verkeer tegen te gaan.

De tekortkomingen van parkeertarieven als oplossing voor congestie hebben geleid tot wat we nu noemen kilometerheffing (ook bekend als rekeningrijden, anders betalen voor mobiliteit of het heffen van tol). Bij kilometerheffing wordt er een heffing vastgesteld op segmenten van een vervoersnetwerk om gebruikers efficiënt het netwerk op en door het netwerk te leiden. Een tariefschema bepaalt op welke segmenten, hoeveel en wanneer tol er wordt geheven. Er zijn succesvolle voorbeelden te vinden, waar steden door kilometerheffing zijn hersteld van frustrerende verkeerssituaties, waar infrastructuur is gefinancierd en waar sommige externe effecten zijn verminderd, terwijl dat niet het primaire doel is geweest.

Vandaag de dag leiden externe effecten (zoals geluid, verkeersonveiligheid, uitstoot, enzovoort) tot grote zorgen over economische effecten en gezondheid, en

toch werden deze aspecten genegeerd bij het opstellen van een tariefschema voor kilometerheffing. Een succesvolle implementatie van elke vorm van kilometerheffing hangt vooral af van twee factoren: de politiek en de acceptatie door weggebruikers. In de praktijk hebben we kunnen zien dat het politieke debat over welk tariefschema in te voeren en welk extern effect te bestrijden vaak heeft geleid tot tot een “vroege dood” van kilometerheffing. Daar komt bij dat automobilisten kilometerheffing meestal als een extra belasting zien, waardoor ze in opstand komen. Er ontstaat discussie in het parlement, omdat belanghebbenden verschillende belangen hebben, die vaak niet met elkaar te verenigen zijn: anders zou er immers geen discussie zijn. Een belanghebbende kan bijvoorbeeld een tariefschema nastreven dat congestie vermindert, terwijl iemand anders liever een tariefschema heeft dat het milieu schoon en groen houdt. Een tariefschema dat één specifiek extern effect vermindert, kan een ander effect juist versterken. Dit frustreert het debat en leidt vaak toe dat er geen beslissing wordt genomen. Nog ingewikkelder wordt het als een tariefschema binnen een specifiek land / regio leidt tot een efficiënte verdeling van verkeer, maar tot een verslechtering van de verkeerssituatie in een (nabijgelegen) land / regio. Voor een acceptabel tariefschema moeten alle belanghebbenden en landen dus betrokken worden bij het ontwerp ervan. We hebben een tariefschema nodig dat rekening houdt met conflicterende doelen van belanghebbenden, van regio’s en van gebruikers en iedere belanghebbende tevreden stelt.

Speltheorie biedt veelbelovende modellen om tot een dergelijk tariefschema te komen. In dit proefschrift worden modellen ontwikkeld om de problemen op te lossen die ontstaan bij het ontwerpen van een tariefschema voor kilometerheffing door conflicterende doelstellingen van actoren. Eerst onderzoeken we waarom belanghebbenden normaal niet tot een compromis komen. Dit komt doordat het debat tussen beleidsmakers over het tariefschema in het algemeen geen Nash evenwicht (NE) kent. Dit houdt in dat bij een bepaald tariefschema, en dus een bepaalde verkeerssituatie, er altijd een belanghebbende (of regio) te vinden is die zijn situatie kan verbeteren door een ander tariefschema te kiezen uit de set van alle mogelijke tarieven. Ook al kan het begrenzen van de tarieven NE soms afdwingen, het komt ook voor dat er een cyclisch spel wordt gecreëerd tussen de belanghebbenden, wat leidt tot een spel in een niet-Nash evenwicht. Dit betekent dat ongecoördineerde belanghebbenden of regio’s een onverzadigbaar of niet optimaal tariefschema kunnen creëren, wat leidt tot een instabiele situatie. Het ontwikkelde model heeft als input het verkeersnetwerk, de afwikkelingsvoorwaarden en de doelen van de belanghebbenden en weggebruikers. Op basis daarvan presenteert het een stabiel tariefschema aan de belanghebbenden, dat elk van hen tevreden stelt. De modellen zoals ontwikkeld in hoofdstuk 3 definiëren stabiele tariefschema’s voor belanghebbenden, maar garanderen het bestaan van Nash evenwicht niet.

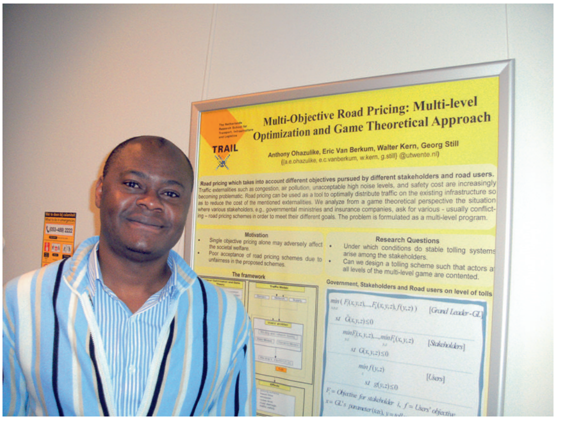
Omdat we onder normale omstandigheden NE tussen de belanghebbenden niet kunnen garanderen, en als het bestaat het NE niet Pareto optimaal hoeft te zijn, hebben we een mechanisme ontworpen dat NE tussen de belanghebbenden introduceert. Dit mechanisme zorgt ervoor dat het uit NE afgeleide tariefschema optimaal is voor alle belanghebbenden en daarnaast Pareto optimaal is voor het systeem als geheel. Dit mechanisme noemen we het optimaal Nash veroorzakend mechanisme. Dus als kilometerheffing zich aandient, ontwerpt het mechanisme een tariefschema voor het gegeven netwerk en presenteert de belanghebbenden een tariefschema dat resulteert in een Nash evenwicht. Op deze manier kunnen belanghebbenden (of regio's) het tariefschema niet voor zichzelf verbeteren, wat waarschijnlijk tot een zekere tevredenheid leidt. Dit nieuwe model en mechanisme voorkomt frustrerende debatten om tot een compromis te komen, terwijl we weten dat een compromisoplossing niet tot een evenwicht hoeft te leiden. Doordat het uit NE afgeleide tariefschema optimaal is voor de belanghebbenden en Pareto optimaal voor het systeem als geheel, maakt het meer waarschijnlijk dat het schema door de belanghebbenden en regio's wordt geaccepteerd. Daarnaast hebben we het belang van de weggebruikers meegenomen in de vorm van lage tarieven, het bestaan van tolvrije wegen en billijkheid (zoals verschillende tarieven voor verschillende inkomensklassen), om zeker te zijn van acceptatie door weggebruikers.

Het optimaal Nash veroorzakend mechanisme kan op veel praktijkproblemen worden toegepast, zoals het afleiden van optimale prestatie in de volgende situaties: 1. Met sensoren uitgeruste auto's wisselend bijvoorbeeld verkeers- en milieu-informatie uit in een telecommunicatienetwerk binnen een beperkt tijdsvenster. 2. Verschillende lokale overheden bepalen toltarieven op aparte regio's in een verkeersnetwerk. 3. Energieproducenten in een energiemarkt die wordt geliberaliseerd. 4. Spel tussen opdrachtgever en opdrachtnemer. 5. Internet providers bij het vaststellen van de prijs van hun diensten. 6. Concurrentie tussen bedrijven binnen de zelfde markt. 7. Werknemers die het aantal werkdagen flexibel kunnen indelen. In hoofdstuk 5 wordt nader uitgelegd hoe we het optimaal Nash veroorzakend mechanisme kunnen aanpassen voor deze problemen. Het speltheoretisch model in dit proefschrift heeft geleid tot een nieuwe manier om optimalisatieproblemen met meerdere doelstellingen (MOP's) op te lossen. In hoofdstuk 6 van dit proefschrift wordt genoemd dat het altijd wenselijk is om alle mogelijke oplossingen van een MOP op een rij te zetten, zodat een beslisser geschikte punten kan kiezen (normaal op het Pareto front). De meeste bestaande algoritmen die alle oplossingen op een rij zetten zijn afhankelijk van het principe van Pareto dominatie. Dit principe is de basis van de meeste genetische algoritmes, welke robuust en krachtig zijn gebleken bij het oplossen van MOP's. Als het aantal doelstellingen toeneemt, vinden deze algoritmes het moeilijk om om te gaan met Pareto dominantie en beginnen dus slechter te presteren. Meer specifiek, vanaf meer dan 4 doelstellingen gaan de algoritmes achteruit. We zien elke doelstelling in het MOP als een speler in de speltheoretische benadering uit dit proefschrift. Ondanks dat het speltheoretische Nash model niet garandeert dat alle niet gedomineerde oplossingen worden gevonden, neigt de concurrentie tussen de spelers (waar elke speler zoekt naar de beste oplossing gegeven wat andere spelers doen) naar het vinden van oplossingen dichtbij het Pareto front. Bij ons testprobleem hebben we gevonden dat alle tijdens het spel gevonden oplossingen op het Pareto front liggen of in de buurt ervan, wat de consistentie met de speltheoretische benadering laat zien. Er worden dus goede oplossingen gegenereerd in een vroeg stadium van het spel en dat is zelden het geval bij het gebruik van genetische algoritmes. Verder wordt het spelmechanisme dat we beschrijven niet slechter als het aantal doelstellingen toeneemt en heeft het niks te maken met Pareto dominatie. Dit brengt ons tot de conclusie dat de speltheoretische benadering een veelbelovende methode is om snel (niet gedomineerde) oplossingen te genereren voor problemen met meerdere doelstellingen. Een stap voor vervolgonderzoek is het inbrengen van goede eigenschappen van genetische algoritmes in de speltheoretische benadering. We geloven dat dat een krachtig instrument voor het oplossingen van MOP's zal opleveren.

Bij tijdelijke verstoringen in het wegennet, zoals ongelukken en wegwerkzaamheden kan het zijn dat mensen hun gebruikelijke reispatroon of route aan moeten passen. Dit leidt meestal tot extra kosten voor de weggebruikers. Bij een systeem van kilometerheffing kan dat tot complicaties leiden. Een wegvak gebaseerd tariefschema kan betekenen dat gebruikers nu meer moeten betalen, omdat ze gebruik moeten maken van een (betaald) wegvak dat ze eerder vermeden. Ook in een route of kilometer gebaseerd systeem moeten gebruikers meer betalen, omdat de gebruikers meer kilometers af moeten leggen om hun rit te maken. Een wegvak gebaseerd systeem kan er toe leiden dat een gebruiker geen keus meer heeft, bijvoorbeeld als hij om zijn eigen woonstraat te bereiken nog maar één (betaalde) route beschikbaar heeft, of alleen een nog duurder alternatief heeft. De literatuur laat zien dat dergelijke complicaties en gevoelens van oneerlijkheid ertoe hebben geleid dat betalen voor weggebruik onpopulair is geworden in veel landen en steden, ook al heeft het erg veel potentie. Daarnaast kunnen de kosten van wegvak of route gebaseerde systemen erg groot zijn, waardoor de implementatie van een vorm van kilometerheffing in veel steden er niet van is gekomen. In dit proefschrift hebben we een tariefsysteem ontwikkeld dat niet afhangt van welke link of welke route er tijdens de rit wordt gebruikt, maar afhangt van de herkomst en de bestemming. Het systeem hoeft weinig gegevens te onthouden: alleen de herkomst en de bestemming, en dus niet de trajectorie van de hele rit. Dit lost meteen enkele aspecten rondom privacy op en bespaart kosten doordat er minder dataopslag nodig is. Dit systeem heeft geen tolpoortjes of andere apparatuur op de weg nodig. In plaats daarvan is een klein (GPS achtig) apparaatje nodig in het voertuig dat de herkomst en de bestemming detecteert (analoog aan de Nederlandse OV-chipkaart). Omdat het tolsysteem op basis van herkomst-bestemming (HB) niet afhangt van de gebruikte wegvakken of routes, vervallen de eerdergenoemde bezwaren voor het beprijzen van wegvakken of routes. Dat betekent dat gebruikers niet extra hoeven te betalen in het geval van een verstoring in het netwerk. Het HB gebaseerde systeem reguleert de hoeveelheid verkeer dat over de tijd het netwerk instroomt, waardoor het mogelijk wordt de spits te verbreden. Een nadeel van dit systeem is dat het (in het algemeen) de routekeuze van gebruikers niet optimaliseert.

Concluderend geeft dit proefschrift inzicht in verschillende manieren om de effecten van kilometerheffing te modelleren. Het maakt daarbij gebruik van speltheorie om het speelveld te modelleren tussen de belanghebbenden die logischerwijs betrokken zijn bij rekeningrijden. Daarnaast is het belang van de weggebruikers opgenomen in de kern van het algoritme dat het tariefschema bepaalt, waardoor het minder waarschijnlijk is dat het schema leidt tot afkeer bij het grote publiek. Een eerste basis voor praktische implementatie wordt gelegd doordat dit proefschrift alle noodzakelijke wiskundige modellen verschaft voor verschillende mogelijke tariefschema's bij rekeningrijden.

About the author



Anthony hails from a small town Ozubulu in Anambra State of Eastern Nigeria. In 2005, Anthony earned a BTech. Degree with First Class Honours in Computer Science and Mathematics from Federal University of Technology Owerri, Nigeria. He was decorated with so many academic and excellence awards during and after his studies, including the Prestigious Federal Government Scholarship Award for Excellent Students and Dean's Award. He worked in a commercial bank as a Fund Transfer Officer till August 2007 before he was awarded a scholarship by the University of Twente, the Netherlands to further his studies in Mathematics Applied. In 2009, Anthony earned his MSc. Degree in Applied Mathematics from the University of Twente. During his Master degree, he was a graduate student at a traffic consultancy company, Goudappel Coffeng in the Netherlands for one year where he investigated a dynamic traffic model and developed traffic models for road pricing. From January 2010 to December 2013, he was a PhD Research Fellow at the Chairs of Discrete Mathematics and Mathematical Programming (DMMP) of the Applied Mathematics department and Centre for Transport Studies (CTS) of the Civil Engineering department, both in the University of Twente. During his PhD, Anthony worked on Road Pricing and Vehicular Adhoc Networks projects. Apart from academia, Anthony likes playing football and talking about great scientists ever lived on the surface of the Earth.

Author's publications

List of author's publications in reverse chronological order

1. Ohazulike, A.E., Still, G., Kern, W. and Berkum, E. C. van (2014) *Multi-Objective Road Pricing: A game theoretic approach.* Submitted for European Journal of Operational Research (under review).

2. Schwartz, R.S., Ohazulike, A.E., Sommer, C., Scholten, J., Dressler, F. & Havinga, P.J.M. (2014). *On the applicability of fair and adaptive data dissemination in traffic information systems.* Ad hoc networks. Volume 13, Part B, February 2014, Pages 428-443.

3. Ohazulike, A.E., Still, G., Kern, W. and Berkum, E. C. van (2013) *An origin–destination based road pricing model for static and multi-period traffic assignment problems.* Transportation Research Part E: Logistics and Transportation Review. Volume 58, November 2013, Pages 1-27. ISSN: 1366-5545.

4. Ohazulike, A. E. & Brands, T. (2013). *Multi-objective Optimization of Traffic Externalities using Tolls: A Comparison of Genetic Algorithm with Game Theoretical Approach.* In IEEE (Ed.), Proceedings of IEEE Conference on Evolutionary Computation, Cancún, Mexico, June 20-23 2013. (pp. 2465-2472).

5. Schwartz, R.S. and Ohazulike, A.E. and Sommer, C. and Scholten, J. and Dressler, F. and Havinga, P.J.M. (2012) *Fair and Adaptive Data Dissemination for Traffic Information Systems.* In: Fourth IEEE Vehicular Networking Conference 2012 (IEEE VNC 2012), 14-16 November 2012, Seoul, South Korea. pp. 1-8. IEEE Intelligent Transportation Systems Society. ISBN 978-1-4673-4996-3.

6. Ohazulike, A.E., Still, G., Kern, W. and Berkum, E. C. van (2012) *Analytical Model of Route-Based Pricing for Time Dependent Traffic Assignment.* (online). In Proceedings of TRAIL Beta congress, 30-31st of October 2012, Rotterdam, The Netherlands. (pp. 1-18). Rotterdam: TRAIL.

7. Ohazulike, A.E., Still, G., Kern, W. and Berkum, E. C. van (2012) *Multi-Objective Road Pricing: A Game Theoretic and Multi-Level Optimization Approach.* In 25th European Conference on Operational Research, 8-11 July 2012, Vilnius, Lithuania.

8. Schwartz, R.S. and Ohazulike, A.E. and Scholten, J. (2012) *Achieving Data Utility Fairness in Periodic Dissemination for VANETs.* In: IEEE 75th Vehicular Technology Conference (VTC2012-Spring), 6-9 May 2012, Yokohama, Japan. pp. 1-5. *IEEE Vehicular Technology Society.* ISSN 1550-2252 ISBN 978-1-4673-0989-9.

9. Ohazulike, A.E., Still, G., Kern, W. and Berkum, E. C. van (2012) *Multi-stakeholder road pricing game: solution concepts.* In: Proceedings of the International Conference on Operations Research (ICOR 2012), 12-13 March 2012, Phuket, Thailand. Issues 63, pp. 10-21, pISSN 2010-376X, eISSN 2010-3778. Published in International Journal of Computational and Mathematical Sciences Volume 6,pp. 1-12. World Academy of Science. pISSN 2010-3905, eISSN 2010-3913.

10. Ohazulike, A.E. and Bliemer, M.C.J. and Still, G.J. and Berkum, E.C. (2012) *Multi-objective road pricing: a game theoretic and multi-stakeholder approach.*In: Compendium of Papers of the Transportation Research Board (TRB) 91st Annual Meeting, 22-26 Jan 2012, Washington, DC. pp. 12-0719. Transportation Research Board. Mira Digital Publishing.

11. Schwartz, R.S. and Ohazulike, A.E. and van Dijk, H.W. and Scholten, J. (2011) *Analysis of Utility-Based Data Dissemination Approaches in VANETs*. In: 4th International Symposium on Wireless Vehicular Communications (WIVEC 2011) - VTC 2011 Fall, 5-6 September 2011, San Francisco, CA, USA. pp. 1-5. IEEE Vehicular Technology Society. ISSN 1090-3038 ISBN 978-1-4244-8328-0.

12. Ohazulike, A.E. and Bliemer, M.C.J. and Still, G.J. and Berkum, E.C. (2010) *Multi-objective road pricing: A cooperative and competitive bi-level optimization approach.* In T.P. Alkim & T. Arentze e.a. (Eds.), 11th Trail Congress Connecting People - Integrating Expertise, 23 and 24 November 2010. (on CD-rom). Delft: TRAIL (ISBN 978-90-5584-139-4).

13. Ohazulike, Anthony E. (2009) *Multi-Objective Road Pricing Problem: A Cooperative and Competitive Bi-level Optimization Approach*, Master thesis, University of Twente, the Netherlands.

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Summary

Road traffic externalities such as congestion, high noise levels, emission, accidents, are increasing due to the rise in vehicle ownership. Owing to financial, geographical and/or feasibility constraints, it could not be practically feasible to combat these externalities by expanding infrastructures. This thesis presents a novel and interesting road pricing approaches to deal with these conflicting objectives with multiple actors. Models show that we can induce optimal system performance among competing stakeholders.

About the Author

Anthony Emeka Ohazulike received his Master's degree in Applied Mathematics from the University of Twente, the Netherlands in 2009. From 2010 to 2013, he went on to complete his PhD at both departments of Applied Mathematics and Traffic Engineering, University of Twente.

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